Mathematical Reasoning — Exam 3
MAT 312, Fall 2011 — D. Ivanšić

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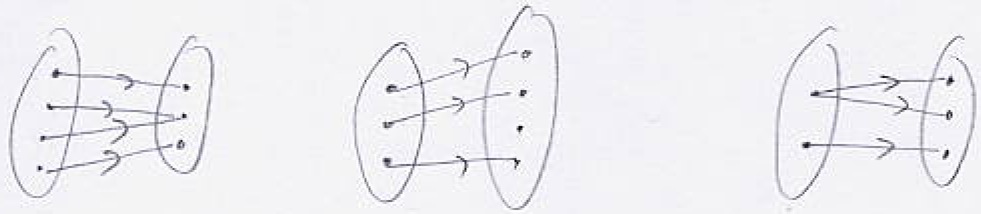
Show all your work!

1. (6pts) Draw arrow diagrams between two sets that illustrate

a) a surjection b) an injection that is not a surjection c) something that is not a function



2. (14pts) Let $\mathbf{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, and let $f : \mathbf{Z}_6 \to \mathbf{Z}_6$, $f(x) = 2x + 1 \pmod 6$.
a) Write the table of function values.
b) What is the set of preimages of 3?
c) What is the set of preimages of 4?
d) Is f injective? Justify.
e) Is f surjective? Justify.

a)

x	$f(x)$	
0	1	
1	3	
2	5	
3	1	$7 \equiv 1 \pmod 6$
4	3	$9 \equiv 3 \pmod 6$
5	5	$11 \equiv 5 \pmod 6$

b) preimages of $3 = \{1, 4\}$

c) preimages of $4 = \emptyset$

d) f is not injective
$f(2) = f(5)$ and $2 \neq 5$

e) f is not surjective;
$\text{range}(f) = \{1, 3, 5\} \neq \text{codomain}(f)$

3. (16pts) Use induction to prove: for every $n \in \mathbf{N}$ and every $x \in \mathbf{R}$, $x \neq 1$,

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

Basis step:

$n = 1$

$1 + x^1 = \frac{x^2 - 1}{x - 1}$ is true, because

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x-1$$

Induction step. Suppose $P(k)$ is true. That is

$$1 + x + x^2 + \dots + x^k = \frac{x^{k+1} - 1}{x - 1} \qquad 1 + x^{k+1}$$

$$1 + x + \dots + x^k + x^{k+1} = \frac{x^{k+1} - 1}{x - 1} + x^{k+1}$$

$$= \frac{x^{k+1} - 1 + (x-1)x^{k+1}}{x-1}$$

$$= \frac{\cancel{x^{k+1}} - 1 + x^{k+2} - \cancel{x^{k+1}}}{x-1}$$

so $1 + x + \dots + x^{k+1} = \frac{x^{k+2} - 1}{x - 1}$

which is $P(k+1)$

4. (18pts) Let $g : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ be given by $g(m, n) = (2m + n, 5m + 3n)$.
a) Evaluate $g(3, -4)$ and $g(-1, -3)$.
b) Find the set of preimages of $(-1, 6)$.
c) Show that g is surjective.

a) $g(3,-4) = (6-4, 15-12)$

$= (2, 3)$

$g(-1,-3) = (-2-3, -5-9)$

$= (-5, -14)$

b) Need (m,n) so that

$g(m,n) = (-1, 6)$

$2m + n = -1$ so $n = -1 - 2m$

$5m + 3n = 6$ $\quad 5m + 3(-1-2m) = 6$

$5m - 3 - 6m = 6$

$-m - 3 = 6$

$m = -9$, so $n = -1 + 18 = 17$

Preimages of $(-1,6)$

$= \{(-9, 17)\}$

c) Given integers s, t, can we solve for m, n:

$\begin{cases} 2m + n = s \\ 5m + 3n = t \end{cases}$ $\quad n = s - 2m$

$5m + 3(s - 2m) = t$

$5m + 3s - 6m = t$

$-m = t - 3s$, $m = 3s - t$

$n = s - 2(3s - t) = s - 6s + 2t = 2t - 5s$

For every (s,t) we have found (m,n) s.t. $g(m,n) = (s,t)$

$m = 3s - t$

$n = 2t - 5s$

5. (18pts) Let $f(x) = \dfrac{4}{x^2+1}$. We assume the codomain is $\mathbf{R}$.
a) What subset of real numbers is the natural domain for this function?
b) Is this function injective? Justify.
c) What is the range of this function? Justify your answer.

a) $x^2+1>0$ for all x, so

$\dfrac{4}{x^2+1}$ is always defined

Domain $= \mathbb{R}$

b) $-1 \neq 1$ and

$$f(-1) = \frac{4}{(-1)^2+1} = 2 = \frac{4}{1^2+1} = f(1)$$

so f is not injective.

c) Need to find for which y the equation

$\dfrac{4}{x^2+1} = y$ has a solution in x;

Since $\dfrac{4}{x^2+1} > 0$ we must have $y>0$

$$\frac{4}{y} = x^2+1$$

$$x^2 = \frac{4}{y} - 1$$

$$x = \pm\sqrt{\frac{4}{y}-1}$$

need to have

$\dfrac{4}{y} - 1 \geq 0 \quad | \cdot y > 0$

$4 - y \geq 0$

$y \leq 4$

Thus, equation has a solution iff

$0 < y \leq 4$, so range $= (0,4]$

6. (8pts) Let $h : \mathbf{R} - \{\frac{1}{3}\} \to \mathbf{R}$ be given by $h(x) = \dfrac{2x}{3x-1}$. Show that h is injective.

Let $x_1, x_2 \neq \frac{1}{3}$, and suppose

$$\frac{2x_1}{3x_1-1} = \frac{2x_2}{3x_2-1} \quad \Big| \cdot (3x_1-1)(3x_2-1)$$

$$2x_1(3x_2-1) = 2x_2(3x_1-1)$$

$$\cancel{6x_1x_2} - 2x_1 = \cancel{6x_1x_2} - 2x_2$$

$$-2x_1 = -2x_2 \quad \Big| \div -2$$

$$x_1 = x_2$$

So h is injective.

7. (20pts) Use induction to prove: when dividing by 3 the sum of squares of three consecutive natural numbers one always gets remainder 2. (For example, dividing $7^2 + 8^2 + 9^2$ by 3 gives remainder 2.)

We need to show that for every $n \in \mathbb{N}$,

$n^2 + (n+1)^2 + (n+2)^2 = 3q + 2$ for some $q \in \mathbb{Z}$

Basis step: $n = 1$

$1^2 + 2^2 + 3^2 = 14 = 3 \cdot 4 + 2$

Induction step: suppose $P(k)$ is true

Then $k^2 + (k+1)^2 + (k+2)^2 = 3q + 2 \quad \Big| + (k+3)^2 - k^2$

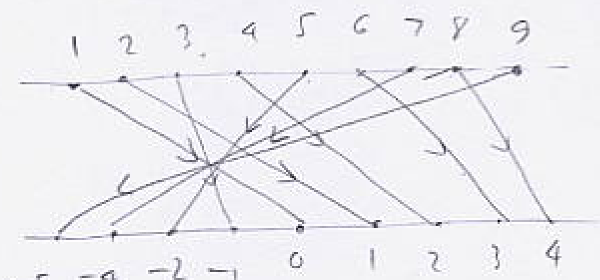
$(k+1)^2 + (k+2)^2 + (k+3)^2 = 3q + 2 + (k+3)^2 - k^2$

$= 3q + 2 + \cancel{k^2} + 6k + 9 - \cancel{k^2}$

$= 3(q + 2k + 3) + 2$

Hence $(k+1)^2 + (k+2)^2 + (k+3)^2$ has form $3q_1 + 2$ for some $q_1 \in \mathbb{N}$, which is $P(k+1)$

Bonus. (10pts) Construct a bijection $\mathbf{N} \to \mathbf{Z}$. Use an arrow diagram to get an idea.



$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$