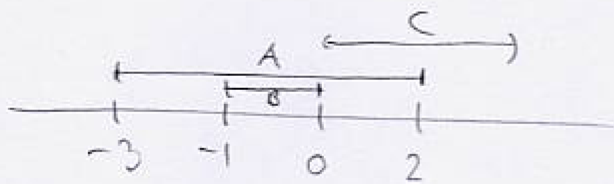
Mathematical Reasoning — Exam 2
MAT 312, Fall 2011 — D. Ivanšić

Name: Solution

Show all your work!

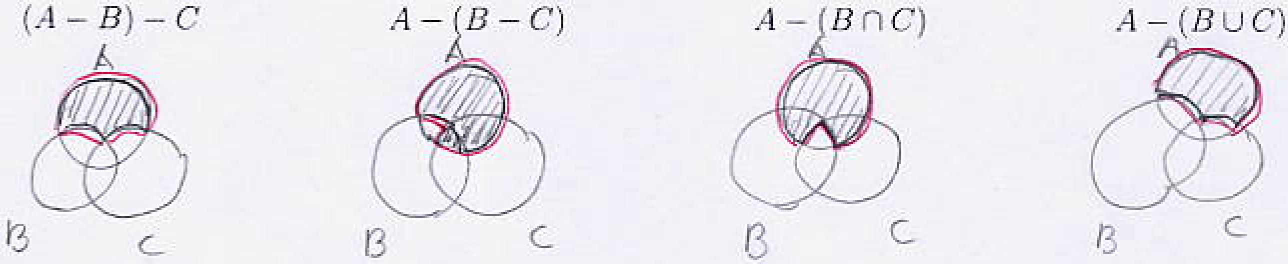
1. (12pts) Let U be the set of real numbers. Consider the intervals $A = [-3, 2]$, $B = [-1, 0]$, $C = (0, \infty)$ and write the following subsets in interval notation (draw the real line if it helps):

$A \cap C$	$A \cup C$	$A - B$	$B - C$	A^c	$A \cap B \cap C$
$(0,2]$	$[-3,\infty)$	$[-3,-1) \cup (0,2]$	$[-1,0]$	$(-\infty,-3) \cup (2,\infty)$	$=\emptyset$ since already $B \cap C = \emptyset$



2. (18pts) Let A, B and C be subsets of some universal set U.
a) Use Venn diagrams to draw the following subsets.
b) Among the four sets, two are equal. Use set algebra to show they are equal.

$(A - B) - C$ $\qquad$ $A - (B - C)$ $\qquad$ $A - (B \cap C)$ $\qquad$ $A - (B \cup C)$



1st and 4th look equal.

$$(A-B)-C = (A \cap B^c) \cap C^c = A \cap (B^c \cap C^c) = A \cap (B \cup C)^c = A - (B \cup C)$$

3. (14pts) Let $A = \{x \in \mathbf{Z} \mid x \equiv 2 \pmod 3\}$ and $B = \{x \in \mathbf{Z} \mid x \equiv 5 \pmod 6\}$.
a) Is $A \subseteq B$? Prove or disprove.
b) Is $B \subseteq A$? Prove or disprove.

$A = \{x \mid x = 3q + 2\}$ $B = \{x \mid x = 6q + 5\}$

$= \{\ldots -4, -1, 2, 5, 8, \ldots\}$ $\{-7, -1, 5, 11, 17, \ldots\}$

a) $A \not\subseteq B$ since $2 \in A$ yet $2 \notin B$

b) Let $x \in B$. Then $6 \mid x - 5$, so there exists a $q \in \mathbb{Z}$ s.t. $x - 5 = 6q$.

Then $x = 6q + 5 = 6q + 3 + 2 = 3(2q + 1) + 2$. Since $2q + 1$ is an integer,

x has form $3q_1 + 2$, so $x \in A$

4. (10pts) Prove: for every real number x, if x is irrational, then $\frac{1}{x}$ is irrational.

Prove by contrapositive: if $\frac{1}{x}$ is rational, then x is rational.

Let x be s.t. $\frac{1}{x}$ is rational. Then $x = \frac{1}{\frac{1}{x}}$. Since

the reciprocal of the rational number $\frac{1}{x}$ is rational,

we conclude $\frac{1}{x}$ is rational.

5. (14pts) Let A, B be subsets of a universal set U. Prove that $A = B$ if and only if $A \cup B = A \cap B$. (Note: one direction can be done simply by set algebra.)

We need to prove two directions:

$\Rightarrow$) If $A=B$, then $A\cup B = A\cap B$.

Let $A=B$.

Then $A\cup B = A\cup A = A = A\cap A = A\cap B$.

$\Leftarrow$) If $A\cup B = A\cap B$, then $A=B$.

Let $A\cup B = A\cap B$.

$\subseteq$) Let $x\in A$. Then $x\in A$ or $x\in B$ is true, so $x\in A\cup B$. Since $A\cup B = A\cap B$, we get $x\in A\cap B$, which means $x\in A$ and $x\in B$. In particular, $x\in B$.

$\supseteq$) The proof is done in exactly the same way.

6. (18pts) Prove the following:
a) For every integer a, if a^3 is divisible by 3, then a is divisible by 3.
b) $\sqrt[3]{9}$ is an irrational number. (Use statement a)).

a) We prove the contrapositive: If $3\nmid a$, then $3\nmid a^3$

Suppose $3\nmid a$. Then $a\equiv 1,2 \pmod 3$

The table shows what a^3 is congruent to.

$a\equiv\square$	$a^3\equiv\square \pmod 3$
1	1
2	$8\equiv 2 \pmod 3$

Since in both cases $a^3\not\equiv 0 \bmod 3$, we have proven the statement.

b) Suppose there is a rational number $\frac{m}{n}$ (m,n reduced) so that $\frac{m}{n}=\sqrt[3]{9}$. Then $\left(\frac{m}{n}\right)^3=9$, that is $m^3=9n^3$

Then $m^3=3(3n^3)$ so $3\mid m^3$.

By a) we get that $3\mid m$, so $m=3k$. It follows that

$(3k)^3=9n^3$

$27k^3=9n^3$

$3k^3=n^3$

This means that $3\mid n^3$, so $3\mid n$. But then 3 divides both m,n, contradicting assumption $\frac{m}{n}$ is reduced.

7. (14pts) Prove that for every real number $a \geq 0$, $a + \dfrac{1}{a} \geq 2$.

Investigation,

$a+\frac{1}{a} \geq 2 \quad | \cdot a$

$a^2+1 \geq 2a$

$a^2-2a+1 \geq 0$

$(a-1)^2 \geq 0$

true, and we can retrace our steps.

Proof: For every number, its square is ≥ 0.

$(a-1)^2 \geq 0$

$a^2-2a+1 \geq 0$

$a^2+1 \geq 2a \quad | \div a$

$a+\frac{1}{a} \geq 2$

Bonus. (10pts) Prove that 131,739,418 is not a square of any integer.

Consider congruences mod 10. Suppose there is an integer a s.t. $a^2 = 131{,}739{,}418$. Then $a \equiv 0,1,2,\ldots,9 \pmod{10}$. Using the table, we find what a^2 is congruent to:

$a \equiv \square$	$a^2 \equiv \square \pmod{10}$
0	0
1	1
2	4
3	9
4	$16 \equiv 6 \pmod{10}$
5	$25 \equiv 5$ "
6	$36 \equiv 6$ "
7	$49 \equiv 9$
8	$64 \equiv 4$
9	$81 \equiv 1$

Thus a^2 must be congruent to one of $0,1,4,5,6,9 \pmod{10}$.

Contradicting $131{,}739{,}418 \equiv 8 \pmod{10}$.

(May also do this in a similar way using congruences mod 4, mod 5, mod 8)