Mathematical Reasoning — Exam 1 MAT 312, Fall 2011 — D. Ivanšić

Name: Solution

Show all your work!

1. (17pts) Consider the following sentences. If a sentence is a statement, determine whether it is true (and justify your answer). If it is a predicate, find its truth set.

a) (universal set=**R**) $x^2 - 5x + 6 = 0$ predicate

$x^2-5x+6=0$
$(x-2)(x-3)=0$
$x=2,3$

Truth set $= \{2,3\}$

b) If an integer is divisible by 4, then it is divisible by 6.

False. Counterexample: 8 is divisible by 4, but not by 6.

c) If $\underbrace{\text{pigs fly}}_{\text{false}}$, then $\underbrace{\text{the sun sets in the east}}_{\text{false}}$.

Statement is true, since false $\Rightarrow$ false is a true statement.

d) There exists a real number x such that $x^2 - 7 = 0$.

$x^2-7=0$
$x^2=7$ $\quad x=\pm\sqrt{7} \in$ real numbers, so the statement is true.

e) For every $x \in \mathbf{R}$, if $x > 1$, then $x^4 > x^2$.

True. Let $x>1$ $\quad | \cdot x$ (positive)
$x^2 > x$

Since $x^2>x$ and $x>1$, we get
$x^2>1 \quad | \cdot x^2$
$x^4>x^2$

2. (8pts) Negate the following statements.

a) Leia does not answer the questions and she suffers the consequences.

Leia answers the questions or she does not suffer the consequences.

b) If Luke goes to the Dagobah system, he gets stuck in the swamp or masters the force.

Luke goes to the Dagobah system and he does not get stuck in the swamp and does not master the force.

3. (8pts) Use a truth table to prove the equivalence $P \iff Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$. (Use however many columns you need.)

and

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$	$\neg P \vee Q$	$P \vee \neg Q$	$(\neg P \vee Q) \wedge (P \vee \neg Q)$
T	T	F	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F	T	F
F	T	T	F	T	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T

same

4. (12pts) Use previously proven logical equivalences to prove the equivalence $P \Longrightarrow (Q \Longrightarrow R) \equiv (P \wedge Q) \Longrightarrow R$. Do not use a truth table.

$$
\begin{aligned}
P \Rightarrow (Q \Rightarrow R) &\equiv \neg P \vee (Q \Rightarrow R) \\
&\equiv \neg P \vee (\neg Q \vee R) \\
&\equiv \neg P \vee \neg Q \vee R \\
&\equiv \neg(P \wedge Q) \vee R \\
&\equiv (P \wedge Q) \Rightarrow R
\end{aligned}
$$

5. (4pts) Write the converse and contrapositive of the statement: If $x > 0$, then $3^x > 1$.

Converse: If $3^x > 1$, then $x > 0$

Contrapositive: If $3^x \leq 1$, then $x \leq 0$

6. (4pts) Use the roster method to write the set $\{x \in \mathbf{Z} \mid x^2 + 4 < 10\}$.

$x^2 + 4 < 10$

$x^2 < 6$

$\{-2, -1, 0, 1, 2\}$

differ by 4

7. (6pts) Use set builder notation to write the set $\{\ldots, -5, -1, 3, 7, 11, \ldots\}$

$= \{n \in \mathbb{Z} \mid n = 3 + 4q \text{ for some } q \in \mathbb{Z}\}$

8. (14pts) For each of the following statements, do the following:
a) Write the statement using symbols.
b) Write the negation of the statement using symbols.
c) Write the negation of the statement in English.

1) There exist real numbers x and y such that $\sin x + \sin y = 3$.

$(\exists x \in \mathbb{R})(\exists y \in \mathbb{R})(\sin x + \sin y = 3)$

$(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\sin x + \sin y \neq 3)$

For every two real numbers x, y, $\sin x + \sin y \neq 3$

2) There exists a positive real number y such that for every real number x, if $x^2 + y < 7$, then $x^2 + y^2 > 25$.

$(\exists y > 0)(\forall x \in \mathbb{R})(x^2 + y < 7 \Rightarrow x^2 + y^2 > 25)$

$(\forall y > 0)(\exists x \in \mathbb{R})(x^2 + y < 7 \text{ and } x^2 + y^2 \leq 25)$

For every $y > 0$ there exists a real number x such that $x^2 + y < 7$ and $x^2 + y^2 \leq 25$.

9. (12pts) Let **R** be the universal set. The following is a predicate in x:

$$(\exists y \in \mathbf{R})(x^2 - y^2 = 16)$$

a) If $x = 3$, is the statement true?
b) If $x = 7$, is the statement true?
c) Find the truth set (the x's) of the above statement. Write it using interval notation.

a) If $x = 3$:

$(\exists y \in \mathbb{R})(9 - y^2 = 16)$

$9 - y^2 = 16$

$y^2 = -7$

no real solution

∴ so statement is false

b) If $x = 7$

$(\exists y \in \mathbb{R})(49 - y^2 = 16)$

$49 - y^2 = 16$

$y^2 = 33$

$y = \pm\sqrt{33}$

Equation has a real solution, so statement is true

c) $(\exists y \in \mathbb{R})(x^2 - y^2 = 16)$ is true when $x^2 - y^2 = 16$ can be solved for y.

$x^2 - y^2 = 16$

$y^2 = x^2 - 16$

$y = \pm\sqrt{x^2 - 16}$ ← solution is real if $x^2 - 16 \geq 0$

That is $x^2 \geq 16$, or $|x| \geq 4$

Truth set: $(-\infty, -4] \cup [4, \infty)$

10. (14pts) An integer n is called a type-0, type-1 or type-2 integer if it can be written in the form $n = 3k$, $n = 3k + 1$ or $n = 3k + 2$, respectively, for some integer k. Prove that if m is a type-1 integer and n is a type-2 integer, then $m^2 + mn + n^2$ is a type-1 integer. Start with a know-show table if you find it helpful.

Suppose the m is a type-1 integer and n is a type-2 integer.

Then there exist integers k, l such that $m = 3k+1$ and $n = 3l+2$.

Then $m^2 + mn + n^2 = (3k+1)^2 + (3k+1)(3l+2) + (3l+2)^2$

$= 9k^2 + 6k + 1 + 9kl + 3l + 6k + 2 + 9l^2 + 12l + 4$

$= 9k^2 + 6k + 9kl + 3l + 6k + 9l^2 + 12l + 7 \leftarrow 6+1$

$= 3(3k^2 + 2k + 3kl + l + 2k + 3l^2 + 4l + 2) + 1$

$= 3(3k^2 + 3l^2 + 3kl + 4k + 5l + 2)$

Since the expression in the parentheses is an integer, we have that $m^2 + mn + n^2$ has form $3q + 1$ (q an integer), hence is type-1.

Bonus. (10pts) Determine whether the statements 1–3 in problem 8 are true and justify.

1) False.

For every $x, y \in \mathbb{R}$:

$-1 \le \sin x \le 1$

$-1 \le \sin y \le 1$

$-2 \le \sin x + \sin y \le 2$

so $\sin x + \sin y$ can never equal 3.

2) True: take $y = 6$.

If $x^2 + 6 < 7$, then $x^2 < 1$.

However, $x^2 + y^2 = x^2 + 6^2 = x^2 + 36 \ge 36 > 25$

for any x,