Calculus 2 — Exam 1
MAT 308, Fall 2011 — D. Ivanšić

Name: ______________________________

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\dfrac{d}{dt}\,(\sqrt{t}+\sqrt[3]{t})(\sqrt{t}-\sqrt[3]{t}) =$

2. (5pts) $\dfrac{d}{dx}\,(5x^2-4x)e^x =$

3. (8pts) $\dfrac{d}{dz}\,\dfrac{z^2+\sqrt{z}}{z^2-\sqrt{z}} =$

4. (4pts) $\dfrac{d}{dx}\,\dfrac{1}{x^2-3x+1} =$

5. (8pts) $\dfrac{d}{d\theta}\,\dfrac{\cos\theta}{\sin^2\theta} =$

6. (6pts) $\dfrac{d}{dx}\,(x^2+3)\ln(x^2+3) =$

7. (5pts) Let $f(x) = \cos(3x)$. What is $f^{(71)}(x)$, the 71st derivative of f? Justify your answer.

Use L'Hopital's rule to find the following limits:

8. (6pts) $\lim_{x\to\infty} \frac{x^3}{e^x} =$

9. (10pts) $\lim_{x\to\infty} (x^2 + x - 2)^{\frac{1}{x}} =$

Find the following antiderivatives.

10. (7pts) $\displaystyle\int 3x^8 - \frac{1}{1+x^2} + \sqrt[4]{x^{17}} + \pi^4 \, dx =$

11. (3pts) $\displaystyle\int e^{5x-7} \, dx =$

12. (7pts) $\displaystyle\int \frac{x^2+1}{\sqrt{x}} \, dx =$

Use the substitution rule in the following integrals:

13. (7pts) $\displaystyle\int \frac{2x-3}{x^2-3x+1} \, dx =$

14. (10pts) $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos^3 x} \, dx =$

15. (8pts) Find the equation of the tangent line to the curve $y = x^2 + 3x - 10$ at the point $(1, -6)$. Sketch the curve and the tangent line on the same graph.

Bonus. (10pts) The rear inside cover of our book claims that

$$\int \frac{x^2\,dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

Verify this formula. *Hint: it's not about figuring out the way to do the integral.*

Calculus 2 — Exam 2
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Show all your work!

1. (14pts) Consider the region enclosed by the curves $y = x^2 - x$ and $y = -x^2 + x + 12$.
a) Sketch the region.
b) Set up the integral that computes its area. Do not evaluate the integral.

2. (20pts) Consider the region bounded by the curves $y = e^x$ and $y = 1 - x$ and $x = -2$.
a) Find the volume of the solid obtained by rotating this region about the x-axis.
b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.

3. (20pts) Consider the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
a) Find the volume of the solid obtained by rotating this region about the y-axis.
b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.

4. (12pts) The base of a solid is the disk bounded by the circle $x^2 + y^2 = 16$. The cross-sections of the solid perpendicular to the y-axis are rectangles whose height is $1/3$ of their width.
a) Sketch the solid and a typical cross-section.
b) Set up the integral for the volume of the solid. Do not evaluate the integral.

5. (18pts) A tank is in the form of an upright circular cone with base radius 3m and height 15m (vertex is at top). Set up the integral for the work needed to fill this tank with water, assuming the water is raised from base level. Assume $g = 10$ and water density $=$ 1000kg/m^3. Do not evaluate the integral, but do draw copious pictures!

6. (16pts) Consider the function $f(x) = 1 - x^2$ over the interval $[-1, 1]$.
a) What is the average value of the function over the interval?
b) What is the geometric interpretation of average value? Sketch a picture.
c) Verify the conclusion of the Mean Value Theorem for integrals, that is, find values of c in the interval so that $f_{\text{ave}} = f(c)$.

Bonus (10pts) Consider the region from problem 2 again, but now rotate it around the vertical line $x = 3$.
a) Set up the integral for the volume of the resulting solid. Do not evaluate the integral.
b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.

Calculus 2 — Exam 3
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Name: ______________________________

Show all your work!

Find the following integrals:

1. (10pts) $\displaystyle\int x^5 \ln x \, dx =$

2. (15pts) $\displaystyle\int \sin^5 x \cos^4 x \, dx =$

3. (15pts) $\displaystyle\int e^{3x} \cos(5x) \, dx =$

Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. (15pts) $\int x^3\sqrt{x^2-1} =$

5. (15pts) $\int_0^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\, dx =$

Use the method of partial fractions to find the following integrals.

6. (12pts) $\displaystyle\int \frac{3x+18}{(x-1)(x+2)}\,dx =$

7. (18pts) $\displaystyle\int \frac{x^2-13x+6}{(x-5)(x^2+9)}\,dx =$

Bonus (10pts) How do those reduction formulas come about? For example, consider the one that reduces $\int \sin^n x\,dx$ to $\int \sin^{n-2} x\,dx$. Start as follows:

$$\int \sin^n x\,dx = \int \sin^{n-2} x \sin^2 x\,dx = \int \sin^{n-2} x(1-\cos^2 x)\,dx = \dots$$

Continue by splitting the last integral and applying a clever integration by parts on $\int \sin^{n-2} x \cos^2 x\,dx$. Soon you will arrive at the reduction formula for $\int \sin^n x\,dx$.

Calculus 2 — Exam 4 **MAT 308, Fall 2011 — D. Ivanšić**	**Name:** ____________________ *Show all your work!*

Determine whether the following improper integrals converge, and, if so, evaluate them.

1. (8pts) $\displaystyle\int_5^\infty \frac{1}{x^{\frac{7}{9}}}\,dx =$

2. (10pts) $\displaystyle\int_0^\infty e^{-3x}\,dx =$

3. (14pts) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2}\,dx =$

4. (12pts) Use comparison to determine whether the improper integral $\displaystyle\int_0^1 \frac{e^x}{\sqrt{x}}$ converges.

5. (14pts) Find the length of the curve $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ from $x = 1$ to $x = 3$.

6. (20pts) The integral $\int_0^{1.2} \sin x^2\, dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $\sin x^2$ is not expressible using elementary functions.
a) Use a program to find M_{25}, the midpoint rule with 25 subintervals.
b) Use the error estimate $|\text{Error}(M_n)| \leq \frac{K_2(b-a)^3}{24n^2}$ to estimate the error that the midpoint rule makes for $n = 25$.
c) What should n be in order for M_n to give you an error less than 10^{-6}?

7. (22pts) Let $f(x) = \sqrt{x}$.
a) Find the 3rd Taylor polynomial for f centered at $a = 4$.
b) Use your calculator to compute the error $|\sqrt{4.5} - T_3(4.5)|$.
c) Use the error estimate $|f(x)-T_n(x)| \leq K_{n+1}\frac{|x-a|^{n+1}}{(n+1)!}$ to estimate the error $|\sqrt{4.5}-T_3(4.5)|$. Does the actual error satisfy this error estimate?
d) How big should n be if we wish to estimate $\sqrt{4.5}$ using $T_n(4.5)$ with accuracy 10^{-7}?

Bonus (10pts) Redoing problem 6, estimate $\int_0^{1.2} \sin x^2 \, dx$ using the Maclaurin polynomial $T_n(x)$ for $\sin x$ by following these steps.
a) Find an n such that $|\sin x - T_n(x)| \leq \frac{5}{6} \times 10^{-6}$ on the entire interval $[0, 1.44]$.
b) Write $T_n(x)$ for the n you found in a).
c) Find the exact value of $\int_0^{1.2} T_n(x^2) \, dx$ and give its decimal value. (Note: you are integrating $T_n(x^2)$, not $T_n(x)$.)
d) Theory guarantees that your answer in c) approximates $\int_0^{1.2} \sin x^2 \, dx$ with accuracy 10^{-6}. From a machine's point of view, which of the two approaches required fewer calculations?
(Note: $\frac{5}{6}$ and 1.44 appear in a) because $\frac{5}{6} = \frac{1}{1.2-0}$ and $1.44 = 1.2^2$.)

Calculus 2 — Exam 5
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Name: ______________________________

Show all your work!

Find the limits, if they exist.

1. (4pts) $\lim\limits_{n\to\infty} \dfrac{100^n}{n!} =$

2. (8pts) $\lim\limits_{n\to\infty} \sin \dfrac{(2n-1)\pi}{4} =$

3. (12pts) $\lim\limits_{n\to\infty} \left(1 - \dfrac{1}{n}\right)^n =$

4. (10pts) Use the theorem that rhymes with the tool that unlocks doors to find:

$\lim\limits_{n\to\infty} \dfrac{2+\sin n}{n^2+4}$

5. (6pts) Write the series using summation notation:

$$\frac{4}{1} - \frac{8}{1\cdot 2} + \frac{16}{1\cdot 2\cdot 3} - \frac{32}{1\cdot 2\cdot 3\cdot 4} + \cdots =$$

6. (12pts) Justify why the series converges and find its sum.

$$\sum_{n=3}^{\infty} \frac{5\cdot 2^{n-1}}{3^{n+1}} =$$

Determine whether the following series converge and justify your answer.

7. (6pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{n^2-1}$

8. (12pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3n}}{n^3-4n^2+1}$

9. (12pts) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - n^3}{n^4 + 4^n}$

10. (18pts) Consider the series $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$.

a) Show that $f(x) = xe^{-x}$ is decreasing from some point on and positive for $x \geq 0$.
b) Justify why you may use the integral test on this series and apply it to determine whether the series converges. *(Hint: use integration by parts.)*

Bonus. (10pts) For which $p > 0$ does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converge?

Calculus 2 — Exam 6
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Name: ______________________________

Show all your work!

Determine whether the following series converge:

1. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+4n+1}{3^{n+4}}$

2. (10pts) $\displaystyle\sum_{n=3}^{\infty} \frac{4^{3n+1}}{n!}$

3. (10pts) Write $\dfrac{5}{7+3x}$ as a power series and indicate the interval where this expansion is valid (do not check the endpoints of the interval for convergence).

4. (20pts) Find the interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{5^n(n^4+n^2+1)}$. Don't forget to check the endpoints of the interval for convergence.

5. (12pts) Use the alternating series test to show that the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+15}$ converges. Thoroughly check the conditions of the test.

6. (8pts) Use the Maclaurin series for e^x to show that $\dfrac{d}{dx}\, e^x = e^x$.

7. (16pts) The integral $\int_0^1 \sin x^2 \, dx$ cannot be found by antidifferentiation, since the antiderivative of $\sin x^2$ is not expressible using elementary functions. However, we can estimate it using series as follows:

a) Use the known Maclaurin series for $\sin x$ to write the Maclaurin series for $\sin x^2$.

b) Integrate the series to find $\int_0^1 \sin x^2 \, dx$, represented as a series.

c) How many terms of the series in b) would be needed to approximate the integral with accuracy 10^{-3}? Write the corresponding partial sum and simplify it to a fraction. (Recall the error estimate: $|s - s_n| < a_{n+1}$.)

8. (14pts) Find the Taylor series expansion of $f(x) = \dfrac{1}{x}$ about $a = 2$. (Use the the general formula for a Taylor series. Or, apply a tricked-out geometric series.)

Bonus. (10pts) Determine whether the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

Calculus 2 — Exam 7
MAT 308, Fall 2011 — D. Ivanšić

Name:

Show all your work!

1. (8pts) Sketch the points in the plane with polar coordinates:

$$\left(2, \frac{3\pi}{4}\right) \qquad \left(-3, \frac{\pi}{6}\right) \qquad \left(-2, -\frac{5\pi}{4}\right)$$

2. (10pts) Convert (a picture may help):
a) $\left(\sqrt{3}, \dfrac{5\pi}{6}\right)$ from polar to rectangular coordinates
b) $(-5, -5)$ from rectangular to polar coordinates

3. (6pts) Write parametric equations for the circle centered at the origin, radius 3, going counterclockwise, such that $c(0) = (0, 3)$.

4. (12pts) Find the equation of the tangent line to the parametric curve $x = t^2 + t - 2$, $y = t^2 - 2t$ at the point when $t = 3$.

5. (12pts) A particle moves along the path $c(t) = (3 - t^2, 4 + 3t^2)$, for $-2 \leq t \leq 2$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

6. (10pts) Find the speed of the particle at time t if the motion is described by $c(t) = (3 \sin \sqrt{t}, 3 \cos \sqrt{t})$.

7. (15pts) An astroid is a curve given by parametric equations $x = \cos^3 t$, $y = \sin^3 t$. Find the length of one quarter of the astroid, which is traced out for $0 \leq t \leq \frac{\pi}{2}$.

8. (15pts) A shell is fired from the origin so that its position is given by $c(t) = (600t, 300t - 5t^2)$, where length is measured in meters, time in seconds.
a) When does the shell reach its highest point?
b) What is the highest altitude achieved?
c) When does the shell hit the ground?
d) How far did the shell travel from the origin?

9. (12pts) Sketch the graph of the function $r = \frac{1}{2} + \sin\theta$ in cartesian coordinates. Then use the intervals of increase and decrease of that graph to help you sketch the polar curve $r = \frac{1}{2} + \sin\theta$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.

Bonus. (10pts) A particle is traveling along the polar curve $r = \sin\theta$, $\theta \geq 0$, where we also treat θ as time.

a) Sketch the curve.

b) Find the expression for the speed of the particle at time θ. *(Hint: you would know how to do this problem if you had parametric equations for x and y, wouldn't you?)*

Calculus 2 — Final Exam **MAT 308, Fall 2011 — D. Ivanšić**

Name: ______________________

Show all your work!

1. (12pts) Consider the region enclosed by the curves $y = x^2 - 2x - 3$ and $y = x + 7$.
a) Sketch the region.
b) Set up the integral that computes its area. Do not evaluate the integral.

2. (16pts) Consider the region bounded by the curves $y = \sin x$ and $y = x$ and $x = \frac{\pi}{2}$. Recall that $\sin x < x$ for $x > 0$.
a) Find the volume of the solid obtained by rotating this region about the x-axis.
b) Sketch the solid and its typical cross-section.

3. (12pts) Integrate: $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^2 x \cos^3 x \, dx =$

4. (12pts) Use trigonometric substitution to evaluate: $\displaystyle\int \frac{x^2}{x^2+9}\, dx =$

5. (16pts) The integral $\int_0^1 e^{x^2}\, dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of e^{x^2} is not expressible using elementary functions.
a) Write out the expression for M_4 for this particular example, the midpoint rule with 4 subintervals.
b) Use the error estimate $|\text{Error}(M_n)| \leq \frac{K_2(b-a)^3}{24n^2}$ to determine how many subintervals are needed if we wish that M_n gives us an error less than 10^{-3}?

Determine whether the following improper integrals converge, and, if so, evaluate them.

6. (6pts) $\displaystyle\int_2^\infty \frac{1}{\sqrt[5]{x}}\, dx =$

7. (10pts) $\displaystyle\int_0^\infty xe^{-x}\, dx =$

8. (8pts) Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4n}{n^5+2n^3}$ converges.

9. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{7 \cdot 3^{n+1}}{2^{2n-1}} =$$

10. (14pts) Find the interval of convergence for the series $\sum_{n=1}^{\infty} n(x-2)^n$. Don't forget to check the endpoints of the interval for convergence.

11. (16pts) a) Write the series expansion for $\dfrac{1}{1+x}$ and state where it converges.
b) Integrate both sides of the equation in a) from $x = 0$ to $x = \frac{1}{2}$.
c) How many terms of the series on the right side of b) would be needed to compute $\ln \frac{3}{2}$ with accuracy 10^{-2}? Write the corresponding partial sum and simplify it to a fraction. (Recall the error estimate: $|s - s_n| < a_{n+1}$.)

12. (8pts) Convert (a picture may help):
a) $\left(2\sqrt{2}, \dfrac{5\pi}{4}\right)$ from polar to rectangular coordinates
b) $(-3, \sqrt{3})$ from rectangular to polar coordinates

13. (10pts) A particle moves along the path $c(t) = (3 + 2\sin t, -2 - 2\cos t)$, for $0 \le t \le \pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

Bonus. (8pts) Determine whether the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

Bonus. (7pts) A particle is traveling along the polar curve $r = f(\theta)$, $\theta \ge 0$, where we also treat θ as time. Find the general expression for the speed of the particle at time θ. *(Hint: you would know how to do this problem if you had parametric equations for x and y, wouldn't you?)*