Calculus 2 — Final Exam
MAT 308, Fall 2011 — D. Ivanšić

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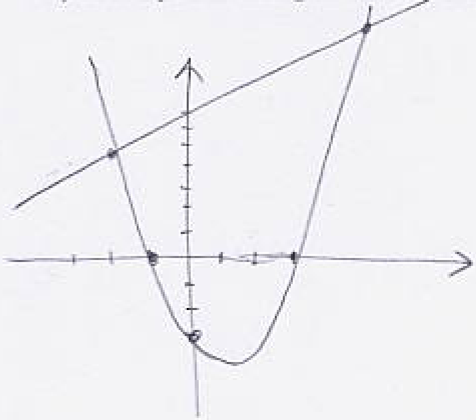
Show all your work!

1. (12pts) Consider the region enclosed by the curves $y = x^2 - 2x - 3$ and $y = x + 7$.
a) Sketch the region.
b) Set up the integral that computes its area. Do not evaluate the integral.



$x^2-2x-3=0$

$(x-3)(x+1)=0$

x-int:

$3, -1$

Intersection:

$x^2-2x-3=x+7$

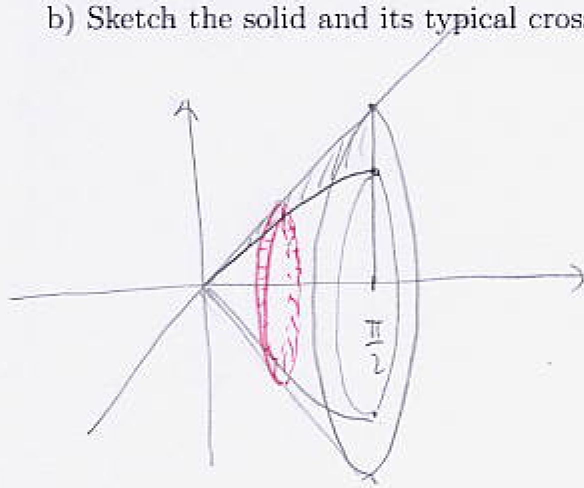
$x^2-3x-10=0$

$(x-5)(x+2)=0$

$x=-2, 5$

$$\text{Area} = \int_{-2}^{5} x+7-(x^2-2x-3)\,dx = \int_{-2}^{5} -x^2+3x+10\,dx$$

2. (16pts) Consider the region bounded by the curves $y = \sin x$ and $y = x$ and $x = \frac{\pi}{2}$. Recall that $\sin x < x$ for $x > 0$.
a) Find the volume of the solid obtained by rotating this region about the x-axis.
b) Sketch the solid and its typical cross-section.



$$V = \int_0^{\pi/2} \pi(x^2 - \sin^2 x)\,dx$$

$$= \pi \int_0^{\pi/2} x^2 - \frac{1-\cos(2x)}{2}\,dx$$

$$= \pi\left(\frac{x^3}{3} - \frac{1}{2}x + \frac{\sin(2x)}{4}\right)\Big|_0^{\pi/2}$$

$$= \pi\left(\frac{1}{3}\left(\left(\frac{\pi}{2}\right)^3 - 0\right) - \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}(\sin\pi - \sin 0)\right)$$

$$= \pi\left(\frac{\pi^3}{24} - \frac{\pi}{4}\right) = \frac{\pi^4 - 6\pi^2}{24}$$

$A = \pi(r_2^2 - r_1^2)$

$r_2 = x$

$r_1 = \sin x$

Determine whether the following improper integrals converge, and, if so, evaluate them.

6. (6pts) $\int_2^\infty \frac{1}{\sqrt[5]{x}}\,dx = \int_2^\infty \frac{1}{x^{1/5}}dx$ p-integral, $p=\frac{1}{5}$, diverges

or: $\int_2^R \frac{1}{x^{1/5}}dx = \frac{5}{4}x^{\frac{4}{5}}\Big|_2^R = \frac{5}{4}\left(R^{4/5}-2^{4/5}\right)$ $\lim_{R\to\infty}\frac{5}{4}\left(R^{4/5}-2^{4/5}\right) = \frac{5}{4}\left(\infty-2^{4/5}\right)$

$= \infty$

7. (10pts) $\int_0^\infty xe^{-x}\,dx =$

$$\int_0^R xe^{-x}dx = \left[\begin{matrix} u=x & v'=e^{-x} \\ u'=1 & v=-e^{-x}\end{matrix}\right] = -xe^{-x}\Big|_0^R - \int_0^R(-e^{-x})\,dx$$

$$= -Re^{-R} + (-e^{-x})\Big|_0^R = -Re^{-R} - (e^{-R}-1)$$

$$\lim_{R\to\infty}\left(1-e^{-R}-Re^{-R}\right) = \lim_{R\to\infty}\left(1-\frac{1}{e^R}-\frac{R}{e^R}\right) = 1-0-\lim_{R\to\infty}\frac{1}{e^R} = 1$$

L'H on third one

converges to 1

8. (8pts) Determine whether the series $\sum_{n=1}^\infty \frac{n^2+4n}{n^5+2n^3}$ converges.

$\frac{n^2+4n}{n^5+2n^3}$ ~~&~~ like $\frac{n^2}{n^5} = \frac{1}{n^3}$.

Do limit comparison test with $\sum\frac{1}{n^3}$!

$$\lim_{n\to\infty}\frac{\frac{n^2+4n}{n^5+2n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty}\frac{\frac{\cancel{n^2}\left(1+\frac{4}{n}\right)}{\cancel{n^5}\,n^3\left(1+\frac{2}{n^2}\right)}}{\cancel{\frac{1}{n^3}}\;1} = \frac{1+0}{1+0} = 1 \neq 0$$

Since $\sum\frac{1}{n^3}$ converges (p-series, $p>1$), so does $\sum\frac{n^2+4n}{n^5+2n^3}$

by limit comparison test.

3. (12pts) Integrate: $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sin^2 x \cos^3 x \, dx = \int_{-\pi/6}^{\pi/6} \sin^2 x \underbrace{\cos^2 x}_{1-\sin^2 x} \cos x \, dx = \left[\begin{array}{ll} u = \sin x & x = \frac{\pi}{6},\ u = \frac{1}{2} \\ du = \cos x \, dx & x = -\frac{\pi}{6},\ u = -\frac{1}{2} \end{array} \right]$

$$= \int_{-1/2}^{1/2} \underbrace{u^2(1-u^2)}_{u^2-u^4} \, du = \left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Bigg|_{-1/2}^{1/2} = \frac{1}{3}\left(\left(\frac{1}{2}\right)^3 - \left(-\frac{1}{2}\right)^3\right) - \frac{1}{5}\left(\left(\frac{1}{2}\right)^5 - \left(-\frac{1}{2}\right)^5\right)$$

$$= \frac{1}{3}\left(2 \cdot \frac{1}{8}\right) - \frac{1}{5}\left(2 \cdot \frac{1}{32}\right) = \frac{1}{12} - \frac{1}{80} = \frac{20-3}{240} = \frac{17}{240}$$

4. (12pts) Use trigonometric substitution to evaluate: $\int \frac{x^2}{x^2+9} \, dx = \left[\begin{array}{l} x = 3\tan\theta \\ dx = 3\sec^2\theta \, d\theta \end{array} \right]$

$$= \int \frac{\tan^2\theta \, 9 \cdot 3\sec^2\theta}{9\tan^2\theta + 9} \, d\theta = \int \frac{3\tan^2\theta \sec^2\theta}{9 \sec^2\theta} \, d\theta = \frac{1}{3}\int \tan^2\theta \, d\theta = \frac{1}{3}\int \sec^2\theta + 1 \, d\theta$$

$$= \frac{1}{3}(\tan\theta + \theta) = \frac{x}{9} + \frac{1}{3}\arctan\frac{x}{3}$$

5. (16pts) The integral $\int_0^1 e^{x^2} \, dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of e^{x^2} is not expressible using elementary functions.

a) Write out the expression for M_4 for this particular example, the midpoint rule with 4 subintervals.

b) Use the error estimate $|\text{Error}(M_n)| \leq \frac{K_2(b-a)^3}{24n^2}$ to determine how many subintervals are needed if we wish that M_n gives us an error less than 10^{-3}?

a) $\frac{1}{8} \quad \frac{3}{8} \quad \frac{5}{8} \quad \frac{7}{8}$

$0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1$

$$M_4 = \frac{1}{4}\left(e^{\left(\frac{1}{8}\right)^2} + e^{\left(\frac{3}{8}\right)^2} + e^{\left(\frac{5}{8}\right)^2} + e^{\left(\frac{7}{8}\right)^2}\right)$$

b) $y = e^{x^2}$

$y' = e^{x^2} \cdot 2x$

$y'' = e^{x^2} \cdot 4x^2 + e^{x^2} \cdot 2$

$= e^{x^2}(4x^2+2)$

On $[0,1]$ $\left|e^{x^2}(4x^2+2)\right| \leq e^1(4 \cdot 1 + 2) \leq 3 \cdot 6 = 18$

May take $K_2 = 18$

Must have: $\frac{18 \cdot (1-0)^3}{24n^2} \leq 10^{-3}$

$\frac{3}{4} \cdot 10^3 \leq n^2$

$750 \leq n^2$

$n \geq \sqrt{750}$

$n \geq 28$

$27^2 = 729$

9. (10pts) Justify why the series converges and find its sum.

$$\sum_{n=1}^{\infty} \frac{7 \cdot 3^{n+1}}{2^{2n-1}} = \sum_{n=1}^{\infty} \frac{7 \cdot 3^n \cdot 3}{2^{2n} \cdot 2^{-1}} = \sum_{n=1}^{\infty} \frac{21}{2^{-1}} \cdot \frac{3^n}{(2^2)^n} = \sum_{n=1}^{\infty} 42 \left(\frac{3}{4}\right)^n = \left[\begin{array}{l} \text{geometric series} \\ r = \frac{3}{4},\ \left|\frac{3}{4}\right| < 1 \end{array}\right]$$

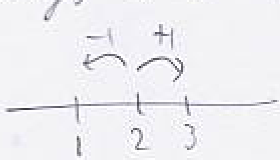
$$= 42 \cdot \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 42 \cdot \frac{\frac{3}{4}}{\frac{1}{4}} = 42 \cdot 3 = 126$$

10. (14pts) Find the interval of convergence for the series $\sum_{n=1}^{\infty} n(x-2)^n$. Don't forget to check the endpoints of the interval for convergence.

Root test:

$$\lim_{n\to\infty} \sqrt[n]{|n(x-2)^n|} = \lim_{n\to\infty} \underbrace{\sqrt[n]{n}}_{\to 1}\, |x-2| = |x-2|$$

Converges when $|x-2| < 1$, that is



when

$1 < x < 3$

For $x = 1$, get $\sum_{n=1}^{\infty} n(-1)^n$

$\lim_{n\to\infty} (-1)^n n \neq 0$

so diverges by test for divergence

For $x = 3$, get $\sum_{n=1}^{\infty} n$

$\lim_{n\to\infty} n = \infty \neq 0$

so series diverges by test for divergence.

Interval of convergence: $(1, 3)$

11. (16pts) a) Write the series expansion for $\dfrac{1}{1+x}$ and state where it converges.
b) Integrate both sides of the equation in a) from $x = 0$ to $x = \frac{1}{2}$.
c) How many terms of the series on the right side of b) would be needed to compute $\ln \frac{3}{2}$ with accuracy 10^{-2}? Write the corresponding partial sum and simplify it to a fraction. (Recall the error estimate: $|s - s_n| < a_{n+1}$.)

a) $$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \quad \Big| \int_0^{1/2}$$

$$\int_0^{1/2} \frac{1}{1+x}\,dx = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} + \cdots\right)\Big|_0^{1/2}$$

$$\ln(1+x)\Big|_0^{1/2} = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} - \frac{\left(\frac{1}{2}\right)^4}{4} + \cdots$$

$$\ln\frac{3}{2} - \ln 1 = \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n\cdot 2^n}$$

Need $\frac{1}{n\cdot 2^n} < 10^{-2}$

$n\cdot 2^n > 100$

n	$n\cdot 2^n$
4	$4\cdot 16 = 64$
5	$5\cdot 32 = 156$

stop for $n = 4$

$$\ln\frac{3}{2} \approx \frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4}$$

$$= \frac{3\cdot 2\cdot 2^4 - 3\cdot 2\cdot 2^2 + 4\cdot 2 - 3}{3\cdot 4\cdot 2^4} = \frac{96 - 24 + 8 - 3}{192}$$

$$= \frac{79}{192}$$

12. (8pts) Convert (a picture may help):
a) $\left(2\sqrt{2}, \dfrac{5\pi}{4}\right)$ from polar to rectangular coordinates
b) $(-3, \sqrt{3})$ from rectangular to polar coordinates

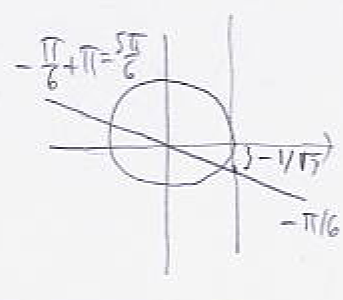
a) $x = 2\sqrt{2}\cos\left(\frac{5\pi}{4}\right) = 2\sqrt{2}\left(-\frac{\sqrt{2}}{2}\right) = -2$

$y = 2\sqrt{2}\sin\left(\frac{5\pi}{4}\right) = 2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 2$

b) $r = \sqrt{(-3)^2 + 3} = \sqrt{12}$

$\tan\theta = \frac{\sqrt{3}}{-3} = -\frac{1}{\sqrt{3}}$ $\quad \theta = \frac{-\pi}{6}$ or $\boxed{\frac{5\pi}{6}}$

since point is in 2nd quadrant



$\left(\sqrt{12}, \frac{5\pi}{6}\right)$

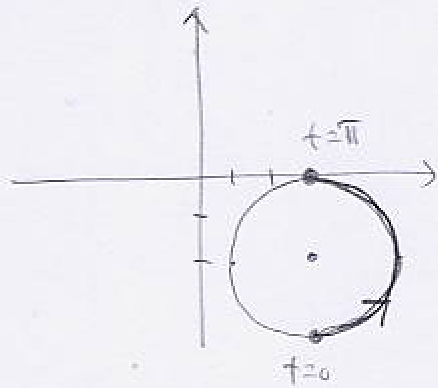
13. (10pts) A particle moves along the path $c(t) = (3 + 2\sin t, -2 - 2\cos t)$, for $0 \le t \le \pi$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

$x = 3 + 2\sin t \qquad x - 3 = 2\sin t$

$y = -2 - 2\cos t \qquad y + 2 = -2\cos t$

$(x-3)^2 + (y+2)^2 = 4\sin^2 t + 4\cos^2 t = 4$

Portion of a circle, centered at $(3,-2)$, radius 2



t	x	y
0	3	-4
π	3	0

Since for smallt, $\sin t > 0$

$3 + 2\sin t > 3$

particle moves as indicated

Bonus. (8pts) Determine whether the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

Root test: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\frac{n^2}{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$ So converges by root test.

Or: $y = \left(\frac{x}{x+1}\right)^x$

$\ln y = x \ln\left(\frac{x}{x+1}\right)$

$\lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\overset{\to \ln 1 = 0}{\ln\left(\frac{x}{x+1}\right)}}{\frac{1}{x} \to 0} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\frac{x}{x+1}} \cdot \frac{1\cdot(x+1) - x\cdot 1}{(x+1)^2}}{-\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} = \lim_{x\to\infty} -\frac{x^2}{x(x+1)} = \lim_{x\to\infty} -\frac{x}{x+1} = -1$

So limit is e^{-1}

Bonus. (7pts) A particle is traveling along the polar curve $r = f(\theta)$, $\theta \ge 0$, where we also treat θ as time. Find the general expression for the speed of the particle at time θ. *(Hint: you would know how to do this problem if you had parametric equations for x and y, wouldn't you?)*

$x = f(\theta)\cos\theta \qquad x' = f'(\theta)\cos\theta - f(\theta)\sin\theta$

$y = f(\theta)\sin\theta \qquad y' = f'(\theta)\sin\theta + f(\theta)\cos\theta$

$\text{speed} = \sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{\left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)^2 + \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)^2}$

$= \sqrt{f'(\theta)^2\cos^2\theta \underbrace{- 2f'(\theta)f(\theta)\sin\theta\cos\theta}_{} + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta \underbrace{+ 2f'(\theta)f(\theta)\sin\theta\cos\theta}_{} + f(\theta)^2\cos^2\theta}$

cancel

$= \sqrt{f'(\theta)^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + f(\theta)^2\underbrace{(\sin^2\theta + \cos^2\theta)}_{=1}} = \sqrt{f'(\theta)^2 + f(\theta)^2}$