Calculus 2 — Exam 7
MAT 308, Fall 2011 — D. Ivanšić

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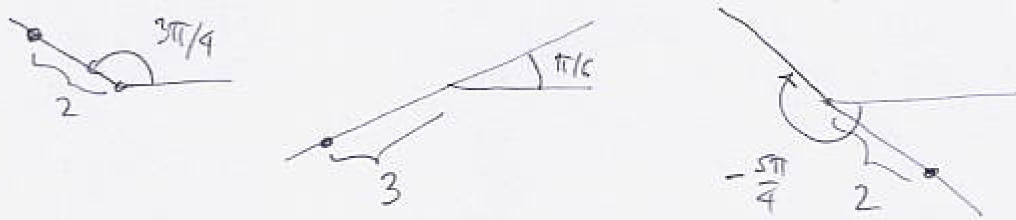
1. (8pts) Sketch the points in the plane with polar coordinates:

$$\left(2, \frac{3\pi}{4}\right) \qquad \left(-3, \frac{\pi}{6}\right) \qquad \left(-2, -\frac{5\pi}{4}\right)$$



2. (10pts) Convert (a picture may help):

a) $\left(\sqrt{3}, \frac{5\pi}{6}\right)$ from polar to rectangular coordinates

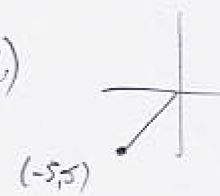
b) $(-5, -5)$ from rectangular to polar coordinates

a) $x = r\cos\theta = \sqrt{3}\cos\frac{5\pi}{6} = \sqrt{3}\cdot\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3}{2}$

$y = r\sin\theta = \sqrt{3}\sin\frac{5\pi}{6} = \sqrt{3}\cdot\frac{1}{2} = \frac{\sqrt{3}}{2}$

$\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$

b)



$\tan\theta = \frac{-5}{-5} = 1$

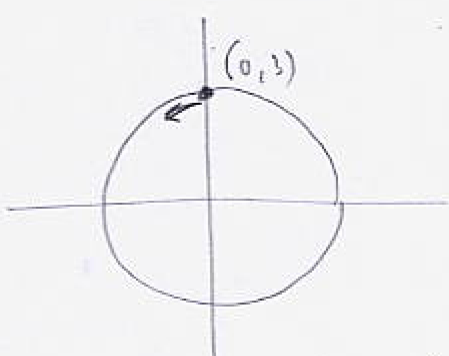
$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

gives correct quadrant

$r = \sqrt{(-5)^2 + (-5)^2} = \sqrt{50} = 5\sqrt{2}$

$\left(5\sqrt{2}, \frac{5\pi}{4}\right)$

3. (6pts) Write parametric equations for the circle centered at the origin, radius 3, going counterclockwise, such that $c(0) = (0, 3)$.



$x = -3\sin t$

$y = 3\cos t$

When $t > 0$ and small

$x < 0,\ y > 0$

15

7. (pts) An astroid is a curve given by parametric equations $x = \cos^3 t$, $y = \sin^3 t$. Find the length of ~~the~~ one quarter of the astroid, which is traced out for $0 \le t \le \frac{\pi}{2}$.

$$l = \int_0^{\pi/2} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{\pi/2} \sqrt{(3\cos^2 t(-\sin t))^2 + (3\sin^2 t \cos t)^2}$$

$$= \int_0^{\pi/2} \sqrt{9(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t)}\, dt = \int_0^{\pi/2} 3\sqrt{\sin^2 t \cos^2 t\, \underbrace{(\cos^2 t + \sin^2 t)}_{=1}}$$

$$= \int_0^{\pi/2} 3 \sin t \cos t\, dt = \int_0^{\pi/2} \frac{3}{2} \sin(2t)\, dt = \frac{3}{2}\left(-\frac{\cos 2t}{2}\right)\Big|_0^{\pi/2} = -\frac{3}{4}(\cos\pi - \cos 0)$$

$$= -\frac{3}{4}\cdot(-2) = \frac{3}{2}$$

15

8. (pts) A shell is fired from the origin so that its position is given by $c(t) = (600t, 300t - 5t^2)$, where length is measured in meters, time in seconds.
a) When does the shell reach its highest point?
b) What is the highest altitude achieved?
c) When does the shell hit the ground?
d) How far did the shell travel from the origin?

a) Need max. of $y(t)$:

$y'(t) = 0$

$300 - 10t = 0$

$t = 30\,s$

b) $y(30) = 300\cdot30 - 5\cdot30^2$

$= 9000 - 4500$

$= 4500\,m$

c) Shell hits ground when $y(t) = 0$

$300t - 5t^2 = 0$

$5t(60 - t) = 0$

$t = 60\,s$

d) $x(60) = 600\cdot60 = 36000\,m$

4. (~~10~~ 12pts) Find the equation of the tangent line to the parametric curve $x = t^2 + t - 2$, $y = t^2 - 2t$ at the point when $t = 3$.

$x' = 2t + 1 \qquad x'(3) = 7$

$y' = 2t - 2 \qquad y'(3) = 4$

$x(3) = 10 \qquad m = \frac{4}{7}$

$y(3) = 3$

Equation is:

$y - 3 = \frac{4}{7}(x - 10)$

$y = \frac{4}{7}x - \frac{40}{7} + 3 = \frac{4}{7}x - \frac{19}{7}$

5. (12pts) A particle moves along the path $c(t) = (3 - t^2, 4 + 3t^2)$, for $-2 \le t \le 2$. Eliminate the parameter in order to sketch the path of motion and then describe the motion of the particle.

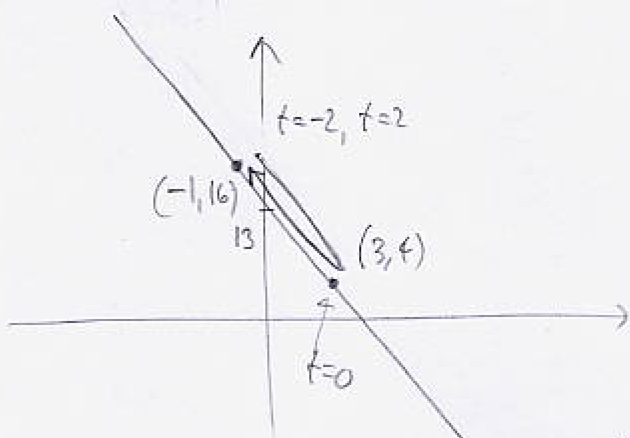
$x = 3 - t^2 \qquad t^2 = 3 - x$

$y = 4 + 3t^2 \qquad t^2 = \frac{y-4}{3}$

$\frac{y-4}{3} = 3 - x$

$y = 4 + 9 - 3x$

$y = -3x + 13$



t	x	y
-2	-1	16
0	3	4
2	-1	16

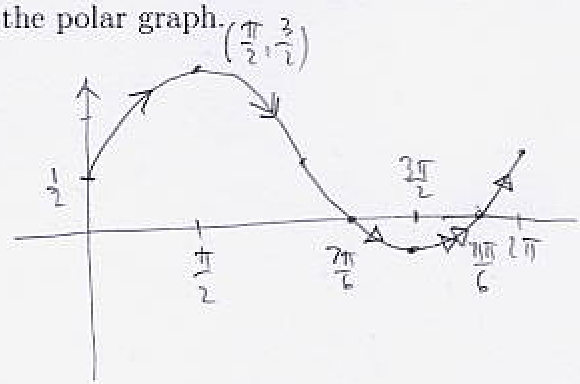
Particle moves once from (-1,16) to (3,4) and back, on line $y = -3x + 13$

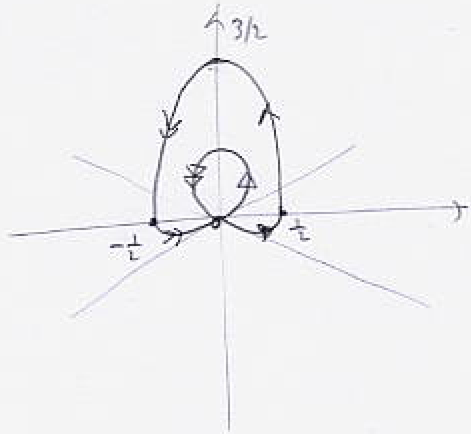
6. (10pts) Find the speed of the particle at time t if the motion is described by $c(t) = (3\sin(\sqrt{t}), 3\cos(\sqrt{t}))$. ~~On what curve is the particle moving?~~

$$\text{speed} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{\left(3\cos\sqrt{t}\cdot\frac{1}{2\sqrt{t}}\right)^2 + \left(-3\sin\sqrt{t}\cdot\frac{1}{2\sqrt{t}}\right)^2}$$

$$= \sqrt{\frac{9}{4t}\underbrace{\left(\cos^2\sqrt{t} + \sin^2\sqrt{t}\right)}_{=1}} = \sqrt{\frac{9}{4t}} = \frac{3}{2\sqrt{t}}$$

9. (12pts) Sketch the graph of the function $r = \frac{1}{2} + \sin\theta$ in cartesian coordinates. Then use the intervals of increase and decrease of that graph to help you sketch the polar curve $r = 3(1 + \sin\theta)$. Indicate which piece of the cartesian graph corresponds to which piece of the polar graph.





$\frac{1}{2} + \sin\theta = 0$

$\sin\theta = -\frac{1}{2}$

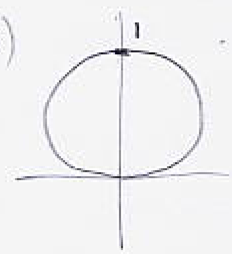
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Bonus. (10pts) A particle is traveling along the polar curve $r = \sin\theta$, $\theta \geq 0$, where we also treat θ as time.

a) Sketch the curve.

b) Find the expression for the speed of the particle at time θ. *(Hint: you would know how to do this problem if you had parametric equations for x and y, wouldn't you?)*

a)



b) $x(\theta) = r(\theta)\cos\theta = \sin\theta\cos\theta$

$y(\theta) = r(\theta)\sin\theta = \sin^2\theta$

$$\text{speed} = \sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(\cos^2\theta - \sin^2\theta)^2 + (2\sin\theta\cos\theta)^2}$$

$$= \sqrt{\cos^4\theta - 2\sin^2\theta\cos^2\theta + \sin^4\theta + 4\sin^2\theta\cos^2\theta}$$

$$= \sqrt{\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta} = \sqrt{(\underbrace{\cos^2\theta + \sin^2\theta}_{=1})^2} = 1$$