Calculus 2 — Exam 6
MAT 308, Fall 2011 — D. Ivanšić

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Show all your work!

Determine whether the following series converge:

1. (10pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+4n+1}{3^{n+4}}$

Root test: $\displaystyle\lim_{n\to\infty} \sqrt[n]{\left|\frac{n^3+4n+1}{3^{n+4}}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n^3+4n+1}}{\sqrt[n]{3^n\cdot 3^4}} = \lim \frac{\sqrt[n]{n^3+4n+1}}{3\cdot\sqrt[n]{3^4}} = \frac{1}{3} < 1$

Converges using root test.

2. (10pts) $\displaystyle\sum_{n=3}^{\infty} \frac{4^{3n+1}}{n!}$

$3n+4-(3n+1)$

Ratio test: $\displaystyle\lim_{n\to\infty} \frac{\frac{4^{3(n+1)+1}}{(n+1)!}}{\frac{4^{3n+1}}{n!}} = \lim_{n\to\infty} \frac{4^{3n+4}}{(n+1)!}\cdot\frac{n!}{4^{3n+1}} = \lim_{n\to\infty} \frac{4^3}{n+1} = 0 < 1$

Converges by root test.

3. (10pts) Write $\dfrac{5}{7+3x}$ as a power series and indicate the interval where this expansion is valid (do not check the endpoints of the interval for convergence).

$$\frac{5}{7+3x} = \frac{5}{7(1+\frac{3}{7}x)} = \frac{5}{7(1-(-\frac{3}{7}x))} = \frac{5}{7}\sum_{n=0}^{\infty}\left(-\frac{3}{7}x\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{5\cdot 3^n}{7^{n+1}}x^n$$

converges when

$\left|-\frac{3}{7}x\right| < 1$ $\quad |x| < \frac{7}{3}$

$\frac{3}{7}|x| < 1$ $\quad -\frac{7}{3} < x < \frac{7}{3}$

4. (20pts) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{5^n(n^4+n^2+1)}$. Don't forget to check the endpoints of the interval for convergence.

Root test $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-3)^n}{5^n(n^4+n^2+1)}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{|x-3|^n}}{\sqrt[n]{5^n}\cdot \underbrace{\sqrt[n]{n^4+n^2+1}}_{\to 1}} = \frac{|x-3|}{5}$

Converges when $\frac{|x-3|}{5} < 1$

which is $|x-3| < 5$

$-5 \quad +5$

$-2 \quad 3 \quad 8$

$x = 8$, get $\sum_{n=1}^{\infty} \frac{5^n}{5^n(n^4+n^2+1)} = \sum \frac{1}{n^4+n^2+1}$

$\frac{1}{n^4+n^2+1} < \frac{1}{n^4}$ since $n^4+n^2+1 > n^4$

Since $\sum \frac{1}{n^4}$ converges, so does $\sum \frac{1}{n^4+n^2+1}$ by comparison test.

$x = -2$

$\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n(n^4+n^2+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+n^2+1}$

Since $\sum \left|\frac{(-1)^n}{n^4+n^2+1}\right|$ converges (at left), this series converges absolutely.

Interval: $[-2, 8]$

5. (12pts) Use the alternating series test to show that the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+15}$ converges. Thoroughly check the conditions of the test.

$\frac{n}{n^2+15}$ decreasing? $\left(\frac{x}{x^2+15}\right)' = \frac{1\cdot(x^2+15) - x\cdot 2x}{(x^2+15)^2} = \frac{15-x^2}{(x^2+15)^2}$

$15 - x^2 = 0$

$x = \pm\sqrt{15}$

$\sqrt{15}$

$f' \quad + \quad 0 \quad -$

so $f(x)$ decreases for $x \geq 4$

$\lim_{n\to\infty} \frac{n}{n^2+15} = \lim_{n\to\infty} \frac{n}{n^2(1+\frac{15}{n^2})} = \lim_{n\to\infty} \frac{1}{n(1+\frac{15}{n^2})} = \frac{1}{\infty\cdot 1} = 0$

Since terms are decreasing and go to 0, Series converges by alternating series test.

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

6. (8pts) Use the MacLaurin series for e^x to show that $\dfrac{d}{dx}e^x = e^x$.

$$\frac{d}{dx}e^x = \frac{d}{dx}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$$

7. (16pts) The integral $\int_0^1 \sin x^2\,dx$ cannot be found by antidifferentiation, since the antiderivative of $\sin x^2$ is not expressible using elementary functions. However, we can estimate it using series as follows:

a) Use the known Maclaurin series for $\sin x$ to write the Maclaurin series for $\sin x^2$.

b) Integrate the series to find $\int_0^1 \sin x^2\,dx$, represented as a series.

c) How many terms of the series in b) would be needed to approximate the integral with accuracy 10^{-3}? Write the corresponding partial sum and simplify it to a fraction. (Recall the error estimate: $|s - s_n| < a_{n+1}$.)

a) $$\sin x^2 = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} + \frac{(x^2)^7}{7!} + \dots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots$$

b) $$\int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots\,dx$$

$$= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \dots \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \dots$$

c) Must have

$$\frac{1}{(2n+1)n!} \le 10^{-3} \quad (\text{odd } n)$$

$$(2n+1)n! \geq 10^3$$

n	$(2n+1)n!$
3	$7\cdot 6 = 42$
5	$120\cdot 11 > 10^3$

$\frac{1}{3} - \frac{1}{7\cdot 6}$ is the approximation

$$= \frac{1}{3} - \frac{1}{42} = \frac{14-1}{42} = \frac{13}{42}$$

8. (14pts) Find the Taylor series expansion of $f(x) = \dfrac{1}{x}$ about $a = 2$. (Use the the general formula for a Taylor series.) ~~What is the interval of convergence for this series (do not check the endpoints)~~?

$y = x^{-1}$

$y' = -1x^{-2}$

$y'' = (-1)(-1)x^{-3}$

$y''' = (-1)(-2)(-3)x^{-4}$

$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5}$

$y^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$

$y^{(n)}(2) = \dfrac{(-1)^n n!}{2^{n+1}}$

$$\frac{1}{x} = \sum_{n=0}^{\infty} \frac{\frac{(-1)^n n!}{2^{n+1}}}{n!}(x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \cdots$$

Using geometric series:

$$\frac{1}{x} = \frac{1}{x-2+2} = \frac{1}{2-(-(x-2))} = \frac{1}{2\left(1-\left(-\frac{x-2}{2}\right)\right)} =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n$$

Bonus. (10pts) Determine whether the series converges.

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$$

Root test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n}{n+1}\right)^{n^2}\right|} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n} =$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

Since $\frac{1}{e} < 1$, series converges by root test.