

Determine whether the following series converge:

1. (10pts) $\sum_{n=1}^{\infty} \frac{n^3 + 4n + 1}{3^{n+4}}$

Root test: $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n^3 + 4n + 1}{3^{n+4}} \right|} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^3 + 4n + 1}}{\sqrt[n]{3^4 \cdot 3^4}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n^3 + 4n + 1}}{3 \cdot \sqrt[n]{3^4}} = \frac{1}{3} < 1$

Converges using root test.

2. (10pts) $\sum_{n=3}^{\infty} \frac{4^{3n+1}}{n!}$

Ratio test: $\lim_{n \rightarrow \infty} \frac{4^{3(n+1)+1}}{(n+1)!} \cdot \frac{n!}{4^{3n+1}} = \lim_{n \rightarrow \infty} \frac{4^{3n+4}}{(n+1)!} \cdot \frac{n!}{4^{3n+1}} = \lim_{n \rightarrow \infty} \frac{4^{3n+4-(3n+1)}}{n+1} = \lim_{n \rightarrow \infty} \frac{4^3}{n+1} = 0 < 1$

Converges by root test.

3. (10pts) Write $\frac{5}{7+3x}$ as a power series and indicate the interval where this expansion is valid (do not check the endpoints of the interval for convergence).

$$\frac{5}{7+3x} = \frac{5}{7(1+\frac{3}{7}x)} = \frac{5}{7(1-(-\frac{3}{7}x))} = \frac{5}{7} \sum_{n=0}^{\infty} \left(-\frac{3}{7}x\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{5 \cdot 3^n}{7^{n+1}} x^n$$

converges when

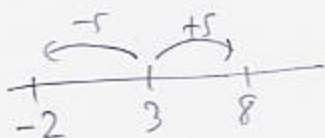
$$\left| -\frac{3}{7}x \right| < 1 \quad |x| < \frac{7}{3}$$

$$\frac{3}{7}|x| < 1 \quad -\frac{7}{3} < x < \frac{7}{3}$$

4. (20pts) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{5^n(n^4+n^2+1)}$. Don't forget to check the endpoints of the interval for convergence.

Root test $\lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{(x-3)^n}{5^n(n^4+n^2+1)} \right|} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{|x-3|^n}}{\sqrt[n]{5^n} \cdot \sqrt[n]{n^4+n^2+1}} = \frac{|x-3|}{5}$

Converges when $\frac{|x-3|}{5} < 1$
 which is $|x-3| < 5$



$x=8$, get $\sum_{n=1}^{\infty} \frac{5^n}{5^n(n^4+n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^4+n^2+1}$

$\frac{1}{n^4+n^2+1} < \frac{1}{n^4}$ since $n^4+n^2+1 > n^4$
 Since $\sum \frac{1}{n^4}$ converges, so does $\sum \frac{1}{n^4+n^2+1}$
 by comparison test.

$x=-2$
 $\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n(n^4+n^2+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4+n^2+1}$

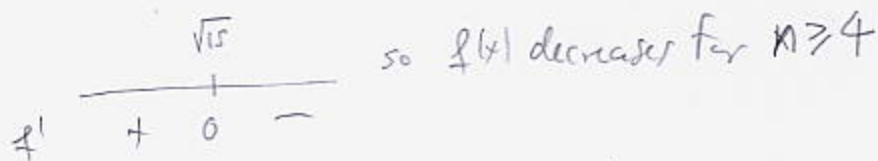
Since $\sum \left| \frac{(-1)^n}{n^4+n^2+1} \right|$ converges (at left),
 this series converges absolutely

Interval: $[-2, 8]$

5. (12pts) Use the alternating series test to show that the series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+15}$ converges.

Thoroughly check the conditions of the test.

$\frac{n}{n^2+15}$ decreasing? $\left(\frac{x}{x^2+15} \right)' = \frac{1 \cdot (x^2+15) - x \cdot 2x}{(x^2+15)^2} = \frac{15-x^2}{(x^2+15)^2}$ $15-x^2=0$
 $x = \pm \sqrt{15}$



$\lim_{n \rightarrow \infty} \frac{n}{n^2+15} = \lim_{n \rightarrow \infty} \frac{n}{n^2(1+\frac{15}{n^2})} = \lim_{n \rightarrow \infty} \frac{1}{n(1+\frac{15}{n^2})} = \frac{1}{\infty \cdot 1} = 0$

Since terms are decreasing and go to 0, series converges by alternating series test.

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

6. (8pts) Use the Maclaurin series for e^x to show that $\frac{d}{dx} e^x = e^x$.

$$\begin{aligned} \frac{d}{dx} e^x &= \frac{d}{dx} \left(1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x \end{aligned}$$

7. (16pts) The integral $\int_0^1 \sin x^2 dx$ cannot be found by antidifferentiation, since the antiderivative of $\sin x^2$ is not expressible using elementary functions. However, we can estimate it using series as follows:

a) Use the known Maclaurin series for $\sin x$ to write the Maclaurin series for $\sin x^2$.

b) Integrate the series to find $\int_0^1 \sin x^2 dx$, represented as a series.

c) How many terms of the series in b) would be needed to approximate the integral with accuracy 10^{-3} ? Write the corresponding partial sum and simplify it to a fraction. (Recall the error estimate: $|s - s_n| < a_{n+1}$.)

$$\begin{aligned} \text{a) } \sin x^2 &= x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots \\ &= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \end{aligned}$$

c) Must have

$$\frac{1}{(2n+1)n!} \leq 10^{-3} \quad (\text{odd } n)$$

$$(2n+1)n! \geq 10^3$$

n	(2n+1)n!
3	7.6 = 42
5	120.11 > 10 ³

$$\text{b) } \int_0^1 x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots dx$$

$$= \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots$$

$$= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \dots$$

$\frac{1}{3} - \frac{1}{7.6}$ is the approximation

$$= \frac{1}{3} - \frac{1}{42} = \frac{14-1}{42} = \frac{13}{42}$$

8. (14pts) Find the Taylor series expansion of $f(x) = \frac{1}{x}$ about $a = 2$. (Use the the general formula for a Taylor series.) ~~What is the interval of convergence for this series (do not check the endpoints)?~~

$$\begin{aligned}
 y &= x^{-1} \\
 y' &= -1x^{-2} \\
 y'' &= (-1)(-2)x^{-3} \\
 y''' &= (-1)(-2)(-3)x^{-4} \\
 y^{(4)} &= (-1)(-2)(-3)(-4)x^{-5} \\
 &\vdots \\
 y^{(n)}(x) &= (-1)^n n! x^{-(n+1)} \\
 y^{(n)}(2) &= \frac{(-1)^n n!}{2^{n+1}}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{x} &= \sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^{n+1}} (x-2)^n \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 + \dots
 \end{aligned}$$

Using geometric series:

$$\begin{aligned}
 \frac{1}{x} &= \frac{1}{x-2+2} = \frac{1}{2 - (-(x-2))} = \frac{1}{2(1 - (-\frac{x-2}{2}))} = \\
 &= \frac{1}{2} \sum_{h=0}^{\infty} \left(-\frac{x-2}{2}\right)^h = \sum_{h=0}^{\infty} \frac{(-1)^h}{2^{h+1}} (x-2)^h
 \end{aligned}$$

Bonus. (10pts) Determine whether the series converges.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

Root test: $\lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n} =$

$$\sim \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}$$

Since $\frac{1}{e} < 1$, series converges by root test.