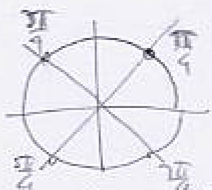
Calculus 2 — Exam 5 MAT 308, Fall 2011 — D. Ivanšić	**Name:** Solution *Show all your work!*

Find the limits, if they exist.

1. (4pts) $\lim_{n\to\infty} \frac{100^n}{n!} = 0$ It is a fact that $\lim_{n\to\infty} \frac{k^n}{n!} = 0$

2. (~~4~~pts) 8 $\lim_{n\to\infty} \sin\frac{(2n-1)\pi}{4} = \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$

5



Limit does not exist.

3. (12pts) $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1} = \frac{1}{e}$

$y = \left(1-\frac{1}{x}\right)^x$

$\ln y = \ln\left(1-\frac{1}{x}\right)^x = x\ln\left(1-\frac{1}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\overset{\to 0}{\ln\left(1-\frac{1}{x}\right)}}{\underset{\to 0}{\frac{1}{x}}} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{1-\left(-\frac{1}{x}\right)}(+x^{-2})}{-x^{-2}}$$

$$= -\frac{1}{1+0} = -1$$

$$\lim_{x\to\infty} y = e^{-1} = \frac{1}{e}$$

4. (10pts) Use the theorem that rhymes with the instrument that unlocks doors to find:

$\lim_{n\to\infty}\frac{2+\sin n}{n^2+4}$

$-1 \le \sin n \le 1$

$1 \le 2+\sin n \le 3$

$$\frac{1}{n^2+4} \le \frac{2+\sin n}{n^2+4} \le \frac{3}{n^2+4}$$

$\lim_{n\to\infty}\frac{1}{n^2+4} = \frac{1}{\infty} = 0$

$\lim_{n\to\infty}\frac{3}{n^2+4} = \frac{3}{\infty} = 0$

equal.

By the squeeze theorem,

$$\lim_{n\to\infty}\frac{2+\sin n}{n^2+4} = 0$$

5. (6pts) Write the series using summation notation:

$$4\,\frac{\cancel{2}}{1} - \frac{8\,\cancel{4}}{1\cdot 2} + \frac{16\,\cancel{8}}{1\cdot 2\cdot 3} - \frac{\cancel{8}32}{1\cdot 2\cdot 3\cdot 4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2^{n+1}}{n!}$$

6. (12 ~~8~~pts) Justify why the series converges and find its sum.

$$\sum_{n=3}^{\infty} \frac{5\cdot 2^{n-1}}{3^{n+1}} = \sum_{n=3}^{\infty} \frac{5\cdot 2^n\cdot 2^{-1}}{3^n\cdot 3} = \sum_{n=3}^{\infty} \frac{5\cdot 2^{-1}}{3}\left(\frac{2}{3}\right)^n = \sum_{n=3}^{\infty} \frac{5}{6}\left(\frac{2}{3}\right)^n = \left[\begin{array}{l}\text{converges,}\\ \text{a geometric series}\\ \text{with } |r| = \left|\frac{5}{6}\right| < 1\end{array}\right]$$

$$= \frac{5}{6}\,\frac{\left(\frac{2}{3}\right)^3}{1-\frac{2}{3}} = \frac{5}{\cancel{6}_{3}}\cdot\frac{\cancel{8}^{4}}{\cancel{27}_{9}}\cdot\frac{\cancel{3}^{1}}{1} = \frac{20}{27}$$

Determine whether the following series converge and justify your answer.

7. (6pts) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+3}{n^2-1}$

$$\lim_{n\to\infty} \frac{n^2+3}{n^2-1} = \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{3}{n^2}\right)}{\cancel{n^2}\left(1-\frac{1}{n^2}\right)} = \frac{1+0}{1-0} = 1$$

Since $\lim_{n\to\infty} a_n \neq 0$ series diverges by divergence test

8. (12 ~~10~~pts) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+3n}}{n^3-4n^2+1}$

like $\sum \frac{\sqrt{n^2}}{n^3} = \sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n^2+3n}}{n^3-4n^2+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n^2\left(1+\frac{3}{n}\right)}}{n^3\left(1-\frac{4}{n}+\frac{1}{n^3}\right)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{\cancel{n}\sqrt{1+\frac{3}{n}}}{\cancel{n^3}_{n^2}\left(1-\frac{4}{n}+\frac{1}{n^3}\right)}}{\cancel{\frac{1}{n^2}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{1+\frac{3}{n}}}{\left(1-\frac{4}{n}+\frac{1}{n^3}\right)} = \frac{\sqrt{1+0}}{1-0+0} = 1 \neq 0$$

Since $\sum \frac{1}{n^2}$ converges (p-series, $p>1$) so does $\sum \frac{\sqrt{n^2+3n}}{n^3-4n^2+1}$ by limit comparison test.

9. (12 pts) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - n^3}{n^4 + 4^n}$

bigger numerator

$$\frac{3^n - n^3}{n^4 + 4^n} \le \frac{3^n}{n^4 + 4^n} \le \frac{3^n}{4^n}$$

bigger denom

Since $\sum \frac{3^n}{4^n} = \sum \left(\frac{3}{4}\right)^n$ converges (geometric series with $r = \frac{3}{4}$, $|r| < 1$)

so does $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - n^3}{n^4 + 4^n}$, by comparison test.

10. (18 pts) Consider the series $\displaystyle\sum_{n=1}^{\infty} n e^{-n}$.

a) Show that $f(x) = xe^{-x}$ is decreasing from some point on and positive for $x \ge 0$.

b) Justify why you may use the integral test on this series and apply it to determine whether the series converges. *(Hint: use integration by parts.)*

a) $f'(x) = 1 \cdot e^{-x} + x \cdot e^{-x}(-1)$

$= \underbrace{e^{-x}}_{>0}(1 - x)$

		1	
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

Since $f'(x) < 0$ for $x > 1$,

$f(x)$ is decreasing for $x \ge 1$,

So the sequence is decreasing for $n \ge 1$

b) by a), $\{ne^{-n}\}$ is decreasing for $n \ge 1$

Also, $ne^{-n} > 0$

These are conditions for the integral test, so we may apply it:

$\displaystyle\int_1^{\infty} xe^{-x}\,dx$: $\displaystyle\int_1^R xe^{-x}\,dx = \begin{bmatrix} u = x & v' = e^{-x} \\ u' = 1 & v = -e^{-x} \end{bmatrix}$

$= -xe^{-x}\Big|_1^R + \displaystyle\int_1^R e^{-x}\,dx = e^{-1} - Re^{-R} + (-e^{-x})\Big|_1^R$

$= e^{-1} - Re^{-R} + e^{-1} - e^{-R} = \frac{2}{e} - \frac{R+1}{e^R}$

$\displaystyle\lim_{R\to\infty} \frac{2}{e} - \frac{R+1}{e^R} = \frac{2}{e} - \lim_{R\to\infty} \frac{R+1}{e^R} \overset{\text{l'H}}{=} \frac{2}{e} - \lim_{R\to\infty} \frac{1}{e^R}$

$= \frac{2}{e} - \frac{1}{\infty} = \frac{2}{e}$

Therefore, series converges by integral test.

Bonus. (10pts) For which $p > 0$ does $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converge?

Converges for $p>1$
Diverges for $0<p\le 1$

When $n \geq e$, then $\ln n \geq 1$

Thus: $\ln n \geq 1 \quad 1 \cdot n^p$

$n^p \ln n \geq n^p$

$\frac{1}{n^p \ln n} \leq \frac{1}{n^p}$ for $n \geq e$

1) If $p>1$, then $\sum \frac{1}{n^p}$ converges (p-series, $p>1$)

Then $\sum \frac{1}{n^p \ln n}$ converges by comparison test.

2) If $p=1$ we get $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$\frac{1}{n \ln n} > 0$ and $\left\{\frac{1}{n \ln n}\right\}$ is decreasing, so we may apply the integral test:

$$\int_2^\infty \frac{1}{x \ln x} dx \longrightarrow \int_2^R \frac{1}{x \ln x} dx = \begin{bmatrix} u = \ln x & x=R,\ u=\ln R \\ du = \frac{1}{x} dx & x=2,\ u = \ln 2 \end{bmatrix} = \int_{\ln 2}^{\ln R} \frac{1}{u} du$$

$$= \ln|u| \Big|_{\ln 2}^{\ln R} = \ln \ln R - \ln \ln 2$$

$\lim_{R \to \infty} (\underbrace{\ln \ln R}_{\to \infty} - \ln \ln 2) = \infty - \ln \ln 2 = \infty$, so integral diverges.

Thus, $\sum \frac{1}{n \ln n}$ diverges by integral test.

3) When $p<1$, $n^p < n$

Then $n^p \ln n < n \ln n$

$\therefore \frac{1}{n^p \ln n} > \frac{1}{n \ln n}$

Since $\sum \frac{1}{n \ln n}$ diverges by 2), so does $\sum \frac{1}{n^p \ln n}$ by comparison test.