Calculus 2 — Exam 4 **MAT 308, Fall 2011 — D. Ivanšić**	**Name:** Solution *Show all your work!*

Determine whether the following improper intergrals converge, and, if so, evaluate them.

1. (8pts) $\int_5^\infty \frac{1}{x^{\frac{7}{9}}}\,dx = \lim_{R\to\infty} \frac{9}{2}\left(R^{2/9}-5^{2/9}\right) = \frac{9}{2}\left(\infty - 5^{2/9}\right) = \infty$ diverges.

$\int_5^R x^{-\frac{7}{9}}\,dx = \frac{9}{2}x^{\frac{2}{9}}\Big|_5^R = \frac{9}{2}\left(R^{2/9}-5^{2/9}\right)$

or: Diverges as a p-integral, $p=\frac{7}{9}<1$.

2. (10pts) $\int_0^\infty e^{-3x}\,dx = \lim_{R\to\infty}\frac{1}{3}\left(1-e^{-3R}\right) = \frac{1}{3}\left(1-e^{-\infty}\right) = \frac{1}{3}(1-0) = \frac{1}{3}$

$\int_0^R e^{-3x}\,dx = \frac{e^{-3x}}{-3}\Big|_0^R = -\frac{1}{3}\left(e^{-3R}-e^{-0}\right) = \frac{1}{3}\left(1-e^{-3R}\right)$

3. (14pts) $\int_2^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{R\to\infty}\left(\frac{1}{\ln 2}-\frac{1}{\ln R}\right) = \frac{1}{\ln 2}-\frac{1}{\infty} = \frac{1}{\ln 2}$

$\int_2^R \frac{1}{x(\ln x)^2}\,dx = \begin{bmatrix} u=\ln x & x=R,\ u=\ln R \\ du=\frac{1}{x}dx & x=2,\ u=\ln 2\end{bmatrix} = \int_{\ln 2}^{\ln R}\frac{1}{u^2}\,du = -\frac{1}{u}\Big|_{\ln 2}^{\ln R}$

$= -\left(\frac{1}{\ln R}-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2}-\frac{1}{\ln R}$

4. (12pts) Use comparison to determine whether the improper integral $\int_0^1 \frac{e^x}{\sqrt{x}}$ converges.

On $[0,1]$ $\quad 1\le e^x\le e$

so $\quad \frac{e^x}{\sqrt{x}} \le \frac{e}{\sqrt{x}}$ $\qquad$ Consider $\int_0^1 \frac{e}{\sqrt{x}}\,dx = e\int_0^1\frac{1}{\sqrt{x}}\,dx$

$\int_R^1 \frac{dx}{\sqrt{x}} = \int_R^1 x^{-\frac{1}{2}}\,dx = 2x^{\frac{1}{2}}\Big|_R^1 = 2(1-\sqrt{R})$

$\lim_{R\to 0+} 2(1-\sqrt{R}) = 2\cdot(1-0) = 2$ $\quad$ so $\int_0^1\frac{1}{\sqrt{x}}\,dx$ converges. By comparison theorem, $\int_0^1 \frac{e^x}{\sqrt{x}}\,dx$ converges also.

5. (14pts) Find the length of the curve $y = \dfrac{x^5}{5} + \dfrac{1}{12x^3}$ from $x = 1$ to $x = 3$.

$$y' = \frac{5x^4}{5} + \frac{1}{12}(-3)x^{-4} = x^4 - \frac{1}{4x^4}$$

$$l = \int_1^3 \sqrt{1 + \left(x^4 - \frac{1}{4x^4}\right)^2}\, dx = \int_1^3 \sqrt{1 + x^8 - \underbrace{2 \cdot x^4 \cdot \frac{1}{4x^4}}_{=\frac{1}{2}} - \frac{1}{16x^8}} = \int_1^3 \sqrt{x^8 + \frac{1}{2} + \frac{1}{16x^8}}$$

$$= \int_1^3 \sqrt{\left(x^4 + \frac{1}{4x^4}\right)^2} = \int_1^3 x^4 + \frac{1}{4}x^{-4}\, dx = \left(\frac{1}{5}x^5 + \frac{1}{4}\frac{x^{-3}}{-3}\right)\Big|_1^3 = \frac{1}{5}(3^5 - 1) - \frac{1}{12}\left(\frac{1}{3^3} - \frac{1}{1}\right)$$

$$= \frac{1}{5} \cdot 242 - \frac{1}{12}\left(\frac{1}{27} - 1\right) = \frac{242}{5} + \frac{1}{12}\,\frac{1-27}{27} = \frac{242}{5} - \frac{13}{\underbrace{12 \cdot 27}_{324}} = \frac{78343}{1620} \approx 48.359877$$

6. (20pts) The integral $\int_0^{1.2} \sin x^2\, dx$ is given. It cannot be found by antidifferentiation, since the antiderivative of $\sin x^2$ is not expressible using elementary functions.
a) Use a program to find M_{25}, the midpoint rule with 25 subintervals.
b) Use the error estimate $|\mathrm{Error}(M_n)| \leq \frac{K_2(b-a)^3}{24n^2}$ to estimate the error that the midpoint rule makes for $n = 25$.
c) What should n be in order for M_n to give you an error less than 10^{-6}?

a) $M_{25} = 0.49608 5635$

$$\frac{5.45(1.2-0)^3}{24 \cdot 25^2} = 6.28 \times 10^{-4}$$

b) Need y''.

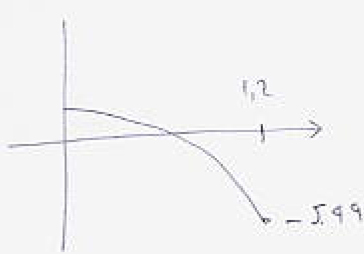
$y = \sin x^2$

$y' = \cos x^2 \cdot 2x = 2x \cos x^2$

$y'' = 2\left(1 \cdot \cos x^2 + x(-\sin x^2) \cdot 2x\right)$

$= 2\left(\cos x^2 - 2x^2 \sin x^2\right)$

Graph is



$\max\limits_{x \in [0,1.2]} |y''| = 5.45$, take $K_2 = 5.45$

c) Need $\dfrac{5.45 \cdot 1.2^3}{24 n^2} \leq 10^{-6} \quad \Big| \cdot 10^6 n^2$

$$\frac{5.45 \cdot 1.2^3 \cdot 10^6}{24} \leq n^2$$

$$n \geq \sqrt{392400} = 626.41$$

Need $n \geq 627$ to guarantee accuracy 10^{-6}.

7. (22pts) Let $f(x) = \sqrt{x}$.

a) Find the 3rd Taylor polynomial for f centered at $a = 4$.

b) Use your calculator to compute the error $|\sqrt{4.5} - T_3(4.5)|$.

c) Use the error estimate $|f(x) - T_n(x)| \le K_{n+1}\frac{|x-a|^{n+1}}{(n+1)!}$ to estimate the error $|\sqrt{4.5} - T_3(4.5)|$. Does the actual error satisfy this error estimate?

d) How big should n be if we wish to estimate $\sqrt{4.5}$ using $T_n(4.5)$ with accuracy 10^{-7}?

a)

n	$y^{(n)}(x)$	$y^{(n)}(4)$	
0	$x^{\frac{1}{2}}$	2	
1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{4}$	
2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{32}$	$4^{-\frac{3}{2}} = \frac{1}{2^3}$
3	$\frac{3}{8}x^{-\frac{5}{2}}$	$\frac{3}{256}$	$4^{-\frac{5}{2}} = \frac{1}{2^5}$
4	$-\frac{15}{16}x^{-\frac{7}{2}}$		

$$T_3(x) = 2 + \frac{\frac{1}{4}}{1!}(x-4) + \frac{-\frac{1}{32}}{2!}(x-4)^2 + \frac{\frac{3}{256}}{3!}(x-4)^3$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

b) $|\sqrt{4.5} - T_3(4.5)| = 1.75 \times 10^{-5}$

c) $K_4 = \max\limits_{x \in [4,4.5]} |y^{(4)}(x)| = \max \left| -\frac{15}{16}\frac{1}{x^{7/2}} \right|$

$= \max\limits_{x \in [4,4.5]} \frac{15}{16}\frac{1}{x^{7/2}}$ ← decreasing function

So max is at 4.

$K_4 = \frac{15}{16} \cdot \frac{1}{4^{7/2}} = \frac{15}{16} \cdot \frac{1}{2^7} = \frac{15}{2048}$

error $\le \dfrac{\frac{15}{2048}(4.5-4)^4}{24} = 1.91 \times 10^{-5}$

yes, actual error $\le 1.91 \times 10^{-5}$

d) From pattern of derivatives and error estimate we see

from $y^{(5)}$ from $|4.5-4|^5$ from $6!$

$K_5 = K_4 \cdot \frac{7}{2} \cdot \frac{1}{4}$ $\quad R_5 = K_5 \cdot \frac{1}{2} \cdot \frac{1}{6}$

Continuing in this way, we get:

n	K_{n+1}
3	1.91×10^{-5}
4	1.39×10^{-6}
5	1.11×10^{-7}
6	$9.6 \times 10^{-9} < 10^{-7}$

Need $n = 6$ to get accuracy 10^{-7}.

$R_6 = R_5 \cdot \frac{9}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{7}$ etc.

Bonus (10pts) Redoing problem 6, estimate $\int_0^{1.2} \sin x^2\,dx$ using the Maclaurin polynomial $T_n(x)$ for $\sin x$ by following these steps.
a) Find an n such that $|\sin x - T_n(x)| \le \frac{5}{6} \times 10^{-6}$ on the entire interval $[0, 1.44]$.
b) Write $T_n(x)$ for the n you found in a).
c) Find the exact value of $\int_0^{1.2} T_n(x^2)\,dx$ and give its decimal value. (Note: you are integrating $T_n(x^2)$, not $T_n(x)$.)
d) Theory guarantees that your answer in c) approximates $\int_0^{1.2} \sin x^2\,dx$ with accuracy 10^{-6}. From a machine's point of view, which of the two approaches required fewer calculations? (Note: $\frac{5}{6}$ and 1.44 appear in a) because $\frac{5}{6} = \frac{1}{1.2-0}$ and $1.44 = 1.2^2$.)

a)

n	$f^{(n)}$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
5	$\cos x$	1

We see that $K_{n+1} \le 1$

Make $1 \cdot \frac{|x-0|^{n+1}}{(n+1)!} \le \frac{5}{6} \times 10^{-6}$

$|x-0| \le |1.44-0|$ so find n so that $\frac{|1.44|^{n+1}}{(n+1)!} \le \frac{5}{6} \times 10^{-6}$

n	$\frac{(1.44)^{n+1}}{(n+1)!}$
5	0.0123
6	0.0025
7	4.58×10^{-4}
8	7.33×10^{-5}
9	1.056×10^{-5}
10	$1.38 \times 10^{-6} \le \frac{5}{6} \times 10^{-6}$
11	1.65×10^{-7}

Take $n = 10$

b) $T_{10}(x) = T_9(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$

$$T_9(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880}$$

c) $$\int_0^{1.2} T_9(x^2)\,dx = \int_0^{1.2} x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \frac{x^{18}}{362880}\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \frac{x^{15}}{75600} + \frac{x^{19}}{6894720}\right)\Bigg|_0^{1.2}$$

$$= \frac{1.2^3}{3} - \frac{1.2^7}{42} + \frac{1.2^{11}}{1320} - \frac{1.2^{15}}{75600} + \frac{1.2^{19}}{6894720}$$

$$= 0.4961158598$$

d) This one has fewer computations – using M_{627} would have 627 evaluations of $\sin x^2$.