Calculus 2 — Exam 3 MAT 308, Fall 2011 — D. Ivanšić

Name: Solution

Show all your work!

Find the following integrals:

1. (10pts) $\int x^5 \ln x\,dx = \begin{bmatrix} u=\ln x & v'=x^5 \\ u'=\frac{1}{x} & v=\frac{x^6}{6} \end{bmatrix} = \frac{x^6}{6}\cdot \ln x - \int \frac{1}{x}\cdot\frac{x^6}{6}\,dx$

$= \frac{x^6}{6}\ln x - \frac{1}{6}\int x^5\,dx = \frac{x^6}{6}\ln x - \frac{x^6}{36} + C$

2. (15pts) $\int \sin^5 x \cos^4 x\,dx = \int \underbrace{\sin^4 x}_{(\sin^2 x)^2=(1-\cos^2 x)^2} \cos^4 x \sin x\,dx = \begin{bmatrix} u=\cos x \\ du=-\sin x\,dx \end{bmatrix}$

$= -\int (1-u^2)^2 u^4\,du$

$= -\int (1-2u^2+u^4)u^4\,du = -\int u^4 - 2u^6 + u^8\,du$

$= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} = -\frac{\cos^5 x}{5} + \frac{2\cos^7 x}{7} - \frac{\cos^9 x}{9} + C$

3. (15pts) $\int e^{3x}\cos(5x)\,dx = \begin{bmatrix} u=e^{3x} & v'=\cos(5x) \\ u'=3e^{3x} & v=\frac{1}{5}\sin(5x) \end{bmatrix}$

$= \frac{1}{5}e^{3x}\sin(5x) - \frac{3}{5}\int e^{3x}\sin(5x)\,dx = \begin{bmatrix} u=e^{3x} & v'=\sin(5x) \\ u'=3e^{3x} & v=-\frac{1}{5}\cos(5x) \end{bmatrix}$

$= \frac{1}{5}e^{3x}\sin(5x) - \frac{3}{5}\left(-\frac{1}{5}e^{3x}\cos(5x) - \left(-\frac{3}{5}\right)\int e^{3x}\cos(5x)\,dx\right)$

$= \frac{1}{5}e^{3x}\sin(5x) + \frac{3}{25}e^{3x}\cos(5x) - \frac{9}{25}\int e^{3x}\cos(5x)\,dx$

Thus: $\frac{34}{25}\int e^{3x}\cos(5x)\,dx = \frac{1}{5}e^{3x}\sin(5x) + \frac{3}{25}e^{3x}\cos(5x) \quad \Big| \cdot \frac{25}{34}$

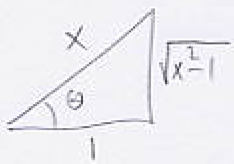
$\int e^{3x}\cos(5x)\,dx = \frac{5}{34}e^{3x}\sin(5x) + \frac{3}{34}e^{3x}\cos(5x)$

Use trigonometric substitution to evaluate the following integrals. Don't forget to return to the original variable where appropriate.

4. ~~5.~~ (15pts) $\int x^3\sqrt{x^2-1}\,dx = \begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\,d\theta \end{bmatrix} = \int \sec^3\theta \underbrace{\sqrt{\sec^2\theta - 1}}_{\tan^2\theta} \sec\theta\tan\theta\,d\theta$

$= \int \sec^4\theta \tan^2\theta\,d\theta = \int \sec^2\theta\tan^2\theta\sec^2\theta\,d\theta = \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \end{bmatrix}$

$= \int (u^2+1)\,u^2\,du = \int u^4 + u^2\,du = \frac{u^5}{5} + \frac{u^3}{3} = \frac{\tan^5\theta}{5} + \frac{\tan^3\theta}{3}$



$x = \sec\theta$

$\frac{1}{x} = \cos\theta$

$= \frac{1}{5}\left(\sqrt{x^2-1}\right)^5 + \frac{1}{3}\left(\sqrt{x^2-1}\right)^3$

$= \frac{1}{5}(x^2-1)^{\frac{5}{2}} + \frac{1}{3}(x^2-1)^{\frac{3}{2}}$

5. ~~6.~~ (15pts) $\int_0^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \begin{bmatrix} x = 2\sin\theta & \sqrt{3} = 2\sin\theta & \theta = \frac{\pi}{3} \\ dx = 2\cos\theta\,d\theta & 0 = 2\sin\theta & \theta = 0 \end{bmatrix}$

$= \int_0^{\pi/3} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \cdot 2\cos\theta\,d\theta = \int_0^{\pi/3} \frac{4\sin^2\theta}{2\cos\theta} \cdot 2\cos\theta\,d\theta = \int_0^{\pi/3} 4\sin^2\theta\,d\theta =$

$4(1-\sin^2\theta)$

$4\cos^2\theta$

$= \int_0^{\pi/3} 4 \cdot \frac{1-\cos(2\theta)}{2}\,d\theta = 2\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/3} = 2\left(\frac{\pi}{3} - \frac{\sin\left(\frac{2\pi}{3}\right)}{2}\right) - 0$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

Use the method of partial fractions to find the following integrals.

6. ~~5~~. (12pts) $\int \frac{3x+18}{(x-1)(x+2)}\,dx = \int \frac{7}{x-1} - \frac{4}{x+2}\,dx = 7\ln|x-1| - 4\ln|x+2| + C$

$$\frac{3x+18}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \quad \Big| \cdot (x-1)(x+2)$$

$$3x+18 = A(x+2) + B(x-1)$$

set: $x=-2$ $\qquad$ $x=1$

$12 = B\cdot(-3)$ $\qquad$ $21 = A\cdot 3$

so $B=-4$ $\qquad$ $A=7$

7. ~~6~~. (18pts) $\int \frac{x^2-13x+6}{(x-5)(x^2+9)}\,dx = \int -\frac{1}{x-5} + \frac{2x-3}{x^2+9}\,dx = -\ln|x-5| + \int \frac{2x}{x^2+9}\,dx - 3\int \frac{dx}{x^2+9}$

$u = x^2+9$

$du = 2x\,dx$

$= -\ln|x-5| + \int \frac{1}{u}\,du - 3\cdot\frac{1}{3}\arctan\frac{x}{3}$

$= -\ln|x-5| + \ln|x^2+9|$

$- \arctan\frac{x}{3} + C$

$$\frac{x^2-13x+6}{(x-5)(x^2+9)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+9} \quad \Big| \cdot (x-5)(x^2+9)$$

$$x^2-13x+6 = A(x^2+9) + (Bx+C)(x-5)$$

Compare coefficients with:

x^0 $\quad 6 = 9A - 5C$

x^1 $\quad -13 = -5B + C$

x^2 $\quad 1 = A + B$

$A=-1$

$B=2$

$C = -13+5B = -3$

put $B = 1-A$ into 2nd equation:

$-13 = -5(1-A) + C$

$-8 = 5A + C \quad |\cdot 5$

$$\begin{cases} -40 = 25A + 5C \\ 6 = 9A - 5C \end{cases}$$

$-34 = 34A$

Bonus (10pts) How do those reduction formulas come about? For example, consider the one that reduces $\int \sin^n x\,dx$ to $\int \sin^{n-2} x\,dx$. Start as follows:

$$\int \sin^n x\,dx = \int \sin^{n-2} x \sin^2 x\,dx = \int \sin^{n-2} x(1-\cos^2 x)\,dx = \dots$$

Continue by splitting the last integral and applying a clever integration by parts on $\int \sin^{n-2} x \cos^2 x\,dx$. Soon you will arrive at the reduction formula for $\int \sin^n x\,dx$.

leave alone

$$= \int \sin^{n-2} x\,dx - \int \sin^{n-2} x \cos^2 x\,dx = \left[\begin{matrix} u = \cos x & v' = \sin^{n-2} x \cos x \\ u' = -\sin x & v = \frac{\sin^{n-1} x}{n-1} \end{matrix}\right]$$

$$= \int \sin^{n-2} x\,dx - \left(\frac{1}{n-1}\sin^{n-1} x \cos x - \left(-\frac{1}{n-1}\right)\int \sin x \sin^{n-1} x\,dx\right), \text{ so}$$

$$\int \sin^n x\,dx = \int \sin^{n-2} x\,dx - \frac{1}{n-1}\sin^{n-1} x \cos x - \frac{1}{n-1}\int \sin^n x\,dx \quad \Big|\, + \frac{1}{n-1}\int \sin^n x\,dx$$

$$\frac{n}{n-1}\int \sin^n x\,dx = \int \sin^{n-2} x\,dx - \frac{1}{n-1}\sin^{n-1} x \cos x \quad \Big|\cdot \frac{n-1}{n}$$

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

Just like the book says.