Calculus 2 — Exam 2 MAT 308, Fall 2011 — D. Ivanšić	**Name:** Solution *Show all your work!*

1. (14pts) Consider the region enclosed by the curves $y = x^2 - x$ and $y = -x^2 + x + 12$.
a) Sketch the region.
b) Set up the integral that computes its area. Do not evaluate the integral.

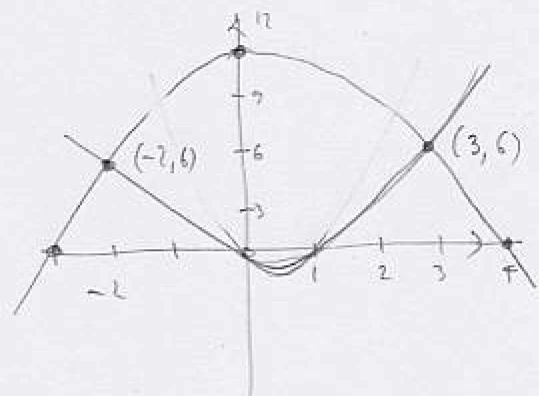
a) x-int. of parabolas:

$x^2 - x = 0 \qquad -x^2 + x + 12 = 0$

$x(x-1) = 0 \qquad x^2 - x - 12 = 0$

$x = 0, 1 \qquad (x-4)(x+3) = 0$

$x = 4, -3$



Intersection:

$x^2 - x = -x^2 + x + 12$

$2x^2 - 2x - 12 = 0 \quad | \div 2$

$x^2 - x - 6 = 0 \qquad x = -2, 3$

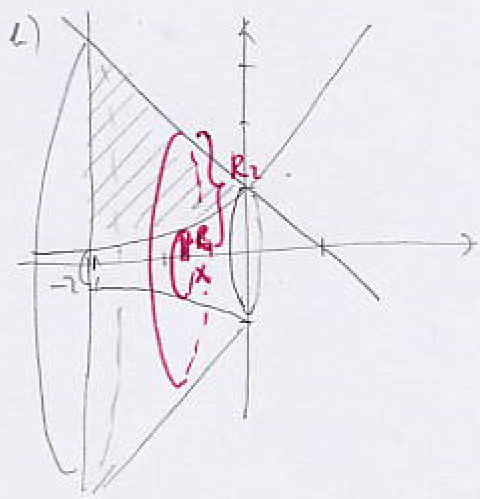
$(x-3)(x+2) = 0$

$$A = \int_{-2}^{3} -x^2 + x + 12 - (x^2 - x)\, dx$$

$$= \int_{-2}^{3} -2x^2 + 2x + 12\, dx$$

2. (20pts) Consider the region bounded by the curves $y = e^x$ and $y = 1 - x$ and $x = -2$.
a) Find the volume of the solid obtained by rotating this region about the x-axis.
b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.

b)



$R_2 = 1 - x$

$R_1 = e^x$

a)
$$V = \int_{-2}^{0} \pi\left((1-x)^2 - (e^x)^2\right) dx$$

$$= \int_{-2}^{0} \pi\left(1 - 2x + x^2 - e^{2x}\right) dx$$

$$= \pi\left(x - x^2 + \frac{x^3}{3} - \frac{e^{2x}}{2}\right)\Big|_{-2}^{0}$$

$$= \pi\left((0-(-2)) - (0^2 - (-2)^2) + \frac{1}{3}(0 - (-2)^3) - \frac{1}{2}(e^0 - e^{-4})\right)$$

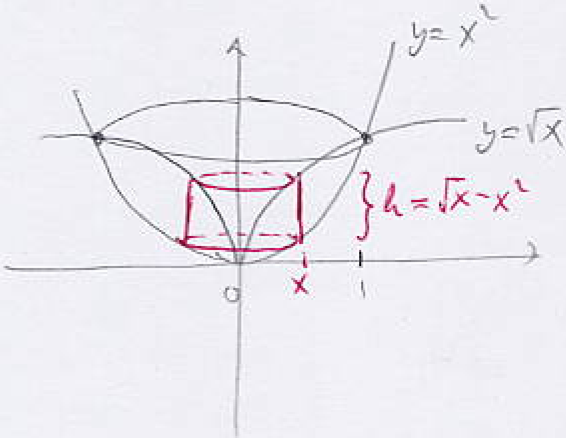
$$= \pi\left(2 + 4 + \frac{8}{3} - \frac{1}{2} + \frac{e^{-4}}{2}\right)$$

$$= \pi\left(\frac{36 + 16 - 3 + 3e^{-4}}{6}\right) = \pi\,\frac{49 + 3e^{-4}}{6}$$

3. (20pts) Consider the region bounded by the curves $y = x^2$ and $y = \sqrt{x}$.
a) Find the volume of the solid obtained by rotating this region about the y-axis.
b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.

Shell method;

b)



$$V = \int_0^1 S(x)\,dx$$

$$= \int_0^1 2\pi x(\sqrt{x} - x^2)\,dx$$

$$= 2\pi \int_0^1 x^{\frac{3}{2}} - x^3\,dx$$

$$= 2\pi\left(\frac{2}{5}x^{\frac{5}{2}} - \frac{x^4}{4}\right)\Big|_0^1$$

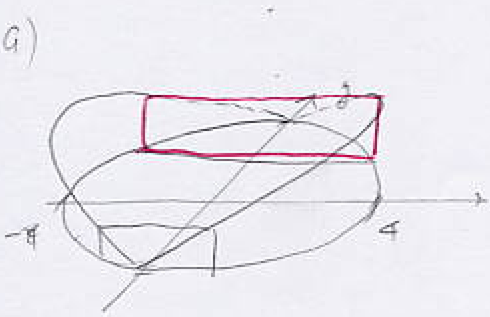
$$S(x) = 2\pi r h$$
$$= 2\pi x \cdot (\sqrt{x} - x^2)$$

$$= 2\pi\left(\frac{2}{5} - \frac{1}{4}\right) = 2\pi\,\frac{8-5}{20}$$

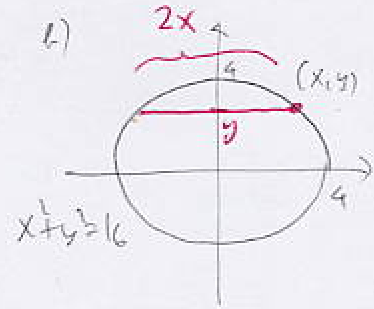
$$= \frac{3\pi}{10}$$

4. (12pts) The base of a solid is the disk bounded by the circle $x^2 + y^2 = 16$. The cross-sections of the solid perpendicular to the y-axis are rectangles whose height is 1/3 of their width.
a) Sketch the solid and a typical cross-section.
b) Set up the integral for the volume of the solid. Do not evaluate the integral.

a)



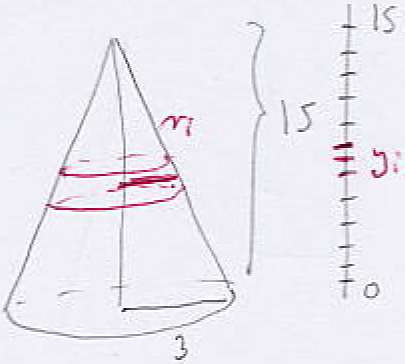
b)



$$A = w \cdot h$$
$$= 2x \cdot \frac{1}{3} \cdot 2x$$
$$= 2\sqrt{16-y^2} \cdot \frac{1}{3}\, 2\sqrt{16-y^2}$$

$$V = \int_{-4}^{4} A(y)\,dy = \int_{-4}^{4} \frac{4}{3}(16 - y^2)\,dy$$

5. (18pts) A tank is in the form of an upright circular cone with base radius 3m and height 15m (vertex is at top). Set up the integral for the work needed to fill this tank with water, assuming the water is raised from base level. Assume $g = 10$ and water density $= 1000\text{kg/m}^3$. Do not evaluate the integral, but do draw copious pictures!



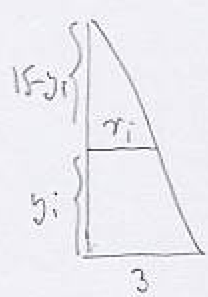
ΔW_i = vol · density · g · dist. pumped

$= \pi r_i^2 \Delta y \cdot 1000 \cdot 10 \cdot y_i$

$= \pi\left(\frac{1}{5}(15-y_i)\right)^2 \Delta y \cdot 10{,}000 \cdot y_i$

$\downarrow$ $n \to \infty$

$W = \int_0^{15} \pi \frac{1}{25}(15-y)^2 \cdot 10000\, y\, dy$



$\frac{r_i}{3} = \frac{15-y_i}{15}$

$r_i = \frac{1}{5}(15-y_i)$

$= \int_0^{15} 400\pi(15-y)^2 y\, dy$

6. (16pts) Consider the function $f(x) = 1 - x^2$ over the interval $[-1, 1]$.

a) What is the average value of the function over the interval?

b) What is the geometric interpretation of average value? Sketch a picture.

c) Verify the conclusion of the Mean Value Theorem for integrals, that is, find values of c in the interval so that $f_{\text{ave}} = f(c)$.

a) $f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{1-(-1)}\int_{-1}^{1}(1-x^2)\,dx$

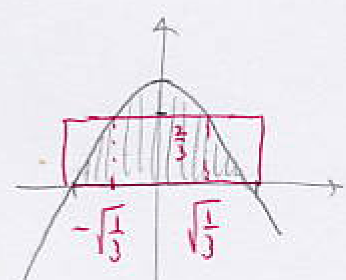
$= \frac{1}{2}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$

$= \frac{1}{2}\left(1-(-1) - \frac{1}{3}(1-(-1))\right)$

$= \frac{1}{2}\left(2 - \frac{2}{3}\right)$

$= \frac{2}{3}$

b)



f_{ave} is the height of a rectangle with base $[-1,1]$ that has the same area as region under curve

c) $1 - x^2 = \frac{2}{3}$

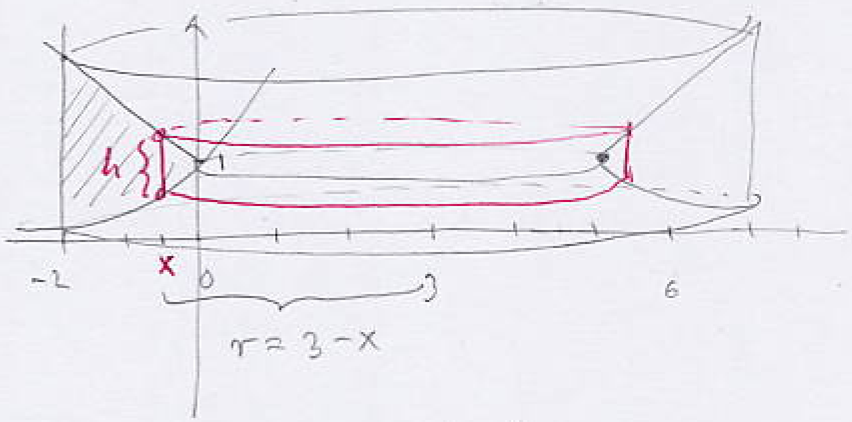
$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}}$

Bonus (10pts) Consider the region from problem 2 again, but now rotate it around the vertical line $x = 3$.

a) Set up the integral for the volume of the resulting solid. Do not evaluate the integral.

b) Sketch the solid and a typical cross-section or cylindrical shell, depending on the method you are using.



$S = 2\pi r h$

$= 2\pi(3-x)(1-x-e^x)$

Shell method better here:

$$V = \int_{-2}^{0} S(x)\,dx = \int_{-2}^{0} 2\pi(3-x)(1-x-e^x)\,dx$$