Calculus 2 — Exam 1 MAT 308, Fall 2011 — D. Ivanšić

Name: Solution

Show all your work!

Differentiate and simplify where appropriate:

1. (6pts) $\frac{d}{dt}(\sqrt{t}+\sqrt[3]{t})(\sqrt{t}-\sqrt[3]{t}) = \frac{d}{dt}\left(\sqrt{t}^{\,2}-\sqrt[3]{t}^{\,2}\right) = \frac{d}{dt}\left(t-t^{\frac{2}{3}}\right) = 1-\frac{2}{3}t^{-\frac{1}{3}}$

2. (5pts) $\frac{d}{dx}(5x^2-4x)e^x = (10x-4)e^x + (5x^2-4x)e^x$

$= e^x(5x^2+6x-4)$

3. (8pts) $\frac{d}{dz}\frac{z^2+\sqrt{z}}{z^2-\sqrt{z}} = \frac{\left(2z+\frac{1}{2\sqrt{z}}\right)(z^2-\sqrt{z})-(z^2+\sqrt{z})\left(2z-\frac{1}{2\sqrt{z}}\right)}{(z^2-\sqrt{z})^2} =$

$= \frac{\cancel{2z^3}+\frac{z^2}{2\sqrt{z}}-2z\sqrt{z}-\cancel{\frac{1}{2}}-\left(\cancel{2z^3}+2z\sqrt{z}-\frac{z^2}{2\sqrt{z}}-\cancel{\frac{1}{2}}\right)}{(z^2-\sqrt{z})^2} = \frac{\frac{z^2}{\sqrt{z}}-4z\sqrt{z}}{(z^2-\sqrt{z})^2}\cdot\frac{\sqrt{z}}{\sqrt{z}} = \frac{-3z^2}{\sqrt{z}(z^2-\sqrt{z})^2}$

4. (4pts) $\frac{d}{dx}\frac{1}{x^2-3x+1} = \frac{d}{dx}(x^2-3x+1)^{-1} = (-1)(x^2-3x+1)^{-2}\cdot(2x-3)$

$= -\frac{2x-3}{(x^2-3x+1)^2}$

5. (8pts) $\frac{d}{d\theta}\frac{\cos\theta}{\sin^2\theta} = \frac{-\sin\theta\cdot\sin^2\theta-\cos\theta\cdot 2\sin\theta\cos\theta}{(\sin^2\theta)^2} = \frac{-\cancel{\sin\theta}(\overset{1-\cos^2\theta}{\sin^2\theta}+2\cos^2\theta)}{\sin^{\cancel{4}\,3}\theta}$

$= -\frac{1+\cos^2\theta}{\sin^3\theta}$

6. (6pts) $\frac{d}{dx}(x^2+3)\ln(x^2+3) = 2x\ln(x^2+3)+(x^2+3)\frac{1}{x^2+3}\cdot 2x$

$= 2x(\ln(x^2+3)+1)$

7. (5pts) Let $f(x) = \cos(3x)$. What is $f^{(71)}(x)$, the 71st derivative of f? Justify your answer.

$71 = \underbrace{4 \cdot 12}+3$

12 cycles of 4 returns derivatives to cos

$f^{(71)}(x) = 3^{71} \sin(3x)$

every time we take a derivative, a 3 jumps out.

3 more derivatives of cos is sin

Use L'Hopital's rule to find the following limits:

8. (6pts) $\lim_{x\to\infty} \frac{x^3}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} = \lim_{x\to\infty} \frac{6x}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = \frac{6}{\infty} = 0$

9. (10pts) $\lim_{x\to\infty} (x^2+x-2)^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty} \ln y} = e^0 = 1$

$y = (x^2+x-2)^{\frac{1}{x}}$

$\ln y = \frac{1}{x} \ln(x^2+x-2)$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(x^2+x-2)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x^2+x-2}\cdot(2x+1)}{1} = \lim_{x\to\infty} \frac{2x+1}{x^2+x-2} =$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{2}{2x+1} = \frac{2}{\infty} = 0$

Find the following antiderivatives.

10. (7pts) $\int 3x^8 - \frac{1}{1+x^2} + \sqrt[4]{x^{17}} + \pi^4\, dx =$ $3\frac{x^9}{9} - \arctan x + \frac{4}{21}x^{\frac{21}{4}} + \pi^4 \cdot x + C$

$x^{\frac{17}{4}}$ a constant

11. (3pts) $\int e^{5x-7}\, dx =$ $\frac{e^{5x-7}}{5} + C$

12. (7pts) $\int \frac{x^2+1}{\sqrt{x}}\, dx =$ $\int \frac{x^2}{\sqrt{x}} + \frac{1}{\sqrt{x}}\, dx = \int x^{\frac{3}{2}} + x^{-\frac{1}{2}}\, dx = \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$

Use the substitution rule in the following integrals:

13. (7pts) $\int \frac{2x-3}{x^2-3x+1}\, dx =$ $\left[\begin{array}{l} u = x^2-3x+1 \\ du = 2x-3 \end{array}\right] = \int \frac{du}{u} = \ln|u|$

$= \ln|x^2-3x+1| + C$

14. (10pts) $\int_0^{\frac{\pi}{6}} \frac{\sin x}{\cos^3 x}\, dx =$ $\left[\begin{array}{ll} u = \cos x & x = \frac{\pi}{6},\ u = \frac{\sqrt{3}}{2} \\ du = -\sin x\, dx & x = 0,\ u = 1 \end{array}\right]$

$-du = \sin x\, dx$

$= \int_1^{\frac{\sqrt{3}}{2}} -\frac{du}{u^3} = \int_{\frac{\sqrt{3}}{2}}^{1} u^{-3}\, du = \left.\frac{u^{-2}}{-2}\right|_{\frac{\sqrt{3}}{2}}^{1} = -\frac{1}{2}\left(1 - \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\right) = -\frac{1}{2}\left(1 - \frac{4}{3}\right)$

$= \frac{1}{6}$

15. (8pts) Find the equation of the tangent line to the curve $y = x^2 + 3x - 10$ at the point $(1, -6)$. Sketch the curve and the tangent line on the same graph.

$y = x^2 + 3x - 10$

$y' = 2x + 3$

$y'(1) = 5$

Equation of tan. line:

$y - (-6) = 5(x-1)$

$y + 6 = 5x - 5$

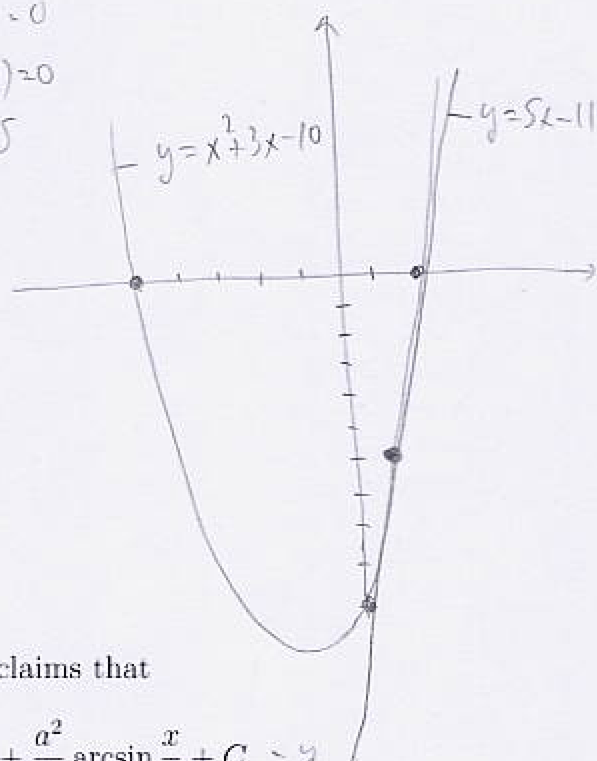
$y = 5x - 11$

Sketch:

$x^2 + 3x - 10 = 0$

$(x+5)(x-2) = 0$

$x = -5, 5$



Bonus. (10pts) The rear inside cover of our book claims that

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C \quad y$$

Verify this formula. *Hint: it's not about figuring out the way to do the integral.*

Taking derivative of right side, we get:

$$y' = -\frac{1}{2}\sqrt{a^2-x^2} - \frac{x}{2}\,\frac{1}{2\sqrt{a^2-x^2}}\cdot(-2x) + \frac{a^2}{2}\,\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a}$$

$$= -\frac{1}{2}\sqrt{a^2-x^2} + \frac{x^2}{2\sqrt{a^2-x^2}} + \frac{a^2}{2}\,\frac{1}{\sqrt{\left(1-\left(\frac{x}{a}\right)^2\right)a^2}} = \frac{1}{2}\left(-\sqrt{a^2-x^2} + \frac{x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{\sqrt{a^2-x^2}}\right)$$

$$= \frac{1}{2}\,\frac{-(a^2-x^2)+x^2+a^2}{\sqrt{a^2-x^2}} = \frac{2x^2}{2\sqrt{a^2-x^2}} = \frac{x^2}{\sqrt{a^2-x^2}}$$

There you have it!