Spring '10/MAT 250/Final Exam **Name:** Solution *Show all your work.*

Find the following limits algebraically, without using L'Hopital's rule.

1. (7pts) $\lim\limits_{x\to 16} \dfrac{\sqrt{x}-4}{x-16} = \lim\limits_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}\cdot\dfrac{\sqrt{x}+4}{\sqrt{x}+4} = \lim\limits_{x\to 16}\dfrac{x-16}{(x-16)(\sqrt{x}+4)} = \lim\limits_{x\to 16}\dfrac{1}{\sqrt{x}+4} = \dfrac{1}{\sqrt{16}+4} = \dfrac{1}{8}$

2. (7pts) $\lim\limits_{x\to\infty} \dfrac{3x^2+2x+17}{x^3-5x^2+7x-1} = \lim\limits_{x\to\infty} \dfrac{x^2\left(3+\underbrace{\frac{2}{x}}_{\to 0}+\underbrace{\frac{17}{x^2}}_{\to 0}\right)}{x^3\left(1-\underbrace{\frac{5}{x}}_{\to 0}+\underbrace{\frac{7}{x^2}}_{\to 0}-\underbrace{\frac{1}{x^3}}_{\to 0}\right)} = \lim\limits_{x\to\infty}\underbrace{\frac{1}{x}}_{\to 0}\cdot 3 = 0\cdot 3 = 0$

3. (8pts) Find the limit. You may use L'Hopital's rule.

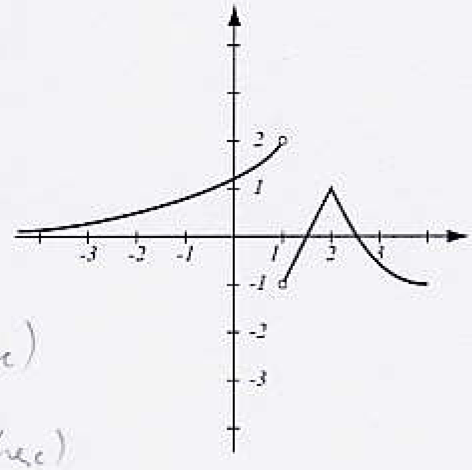
$\lim\limits_{x\to 0} \dfrac{\overset{\to 0}{\tan(5x)}}{\underset{\to 0}{\sin(2x)}} \overset{LH}{=} \lim\limits_{x\to 0}\dfrac{\sec^2(5x)\cdot 5}{\cos(2x)\cdot 2} = \lim\limits_{x\to 0}\dfrac{\frac{1}{\cos^2(5x)}\cdot 5}{\cos(2x)\cdot 2} = \dfrac{\frac{1}{1}\cdot 5}{1\cdot 2} = \dfrac{5}{2}$

Differentiate and simplify where appropriate:

4. (4pts) $\dfrac{d}{dx}\ln(\arctan x) = \dfrac{1}{\arctan x}\cdot\dfrac{1}{1+x^2} = \dfrac{1}{(1+x^2)\arctan x}$

5. (5pts) $\dfrac{d}{d\theta}7^{\sec\theta} = \ln 7\cdot 7^{\sec\theta}\cdot\sec\theta\tan\theta$

6. (11pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.



$\lim_{x\to 1^-} f(x) =$ 2

$\lim_{x\to 1^+} f(x) =$ −1

$\lim_{x\to 1} f(x) =$ d.n.e. (one-sided limits not equal)

List points where f is not continuous (justify).

At $x=1$ (function is not defined there)

List points where f is not differentiable (justify).

At $x=1$ (f is not even continuous there)
$x=2$ (has a sharp point)

7. (16pts) The height of Rogawski's book, t seconds after getting celebratorily thrown upwards with initial velocity 30 meters per second, is given by $h(t) = 30t - 5t^2$ (in meters).
a) Write the expressions for average velocities of the book over two short intervals of time, both beginning with $t = 2$. (Do not evaluate).
b) Use a limit process to find the instantaneous velocity of the flying Rogawski at $t = 2$. Is it going upwards or downwards?

a) Avg. vel. over $[2, 2.1]$ is

$$\frac{h(2.1)-h(2)}{2.1-2} = \frac{30\cdot 2.1 - 5\cdot 2.1^2 - (30\cdot 2 - 5\cdot 2^2)}{2.1-2}$$

$[2, 2.01]$.

$$\frac{h(2.01)-h(2)}{2.01-2} = \frac{30\cdot 2.01 - 5\cdot 2.01^2 - (30\cdot 2 - 5\cdot 2^2)}{2.01-2}$$

b)
$$\lim_{t\to 2} \frac{h(t)-h(2)}{t-2} = \lim_{t\to 2} \frac{30t - 5t^2 - (30\cdot 2 - 5\cdot 4)}{t-2} = \lim_{t\to 2} \frac{-5t^2+30t-40}{t-2}$$

$$= \lim_{t\to 2} \frac{-5(t^2-6t+8)}{t-2} = \lim_{t\to 2} \frac{-5\cancel{(t-2)}(t-4)}{\cancel{t-2}} = \lim_{t\to 2} -5(t-4) = -5(2-4)$$

$$= 10 \text{ m/s}$$

8. (32pts) Let $f(x) = e^{-x}(2x+5)$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. (Use L'Hopital if necessary.)

d) Use information from a)–c) to sketch the graph. (Leave significant y-coordinates in closed form — don't evaluate.)

$$f'(x) = e^{-x}(-1)(2x+5) + e^{-x}\cdot 2$$
$$= e^{-x}(-2x-5+2)$$
$$= e^{-x}(-2x-3)$$

$$f''(x) = e^{-x}(-1)(-2x-3) + e^{-x}(-2)$$
$$= e^{-x}(2x+3-2) = e^{-x}(2x+1)$$

a) $\underbrace{e^{-x}}_{>0}(-2x-3) = 0$

$-2x-3=0$

$x = -\frac{3}{2}$

		$-\frac{3}{2}$	
$f'(x)$	+	0	−
$f(x)$	$\nearrow$	loc. max	$\searrow$

b) $\underbrace{e^{-x}}_{>0}(2x+1) = 0$

$2x+1=0$

$x = -\frac{1}{2}$

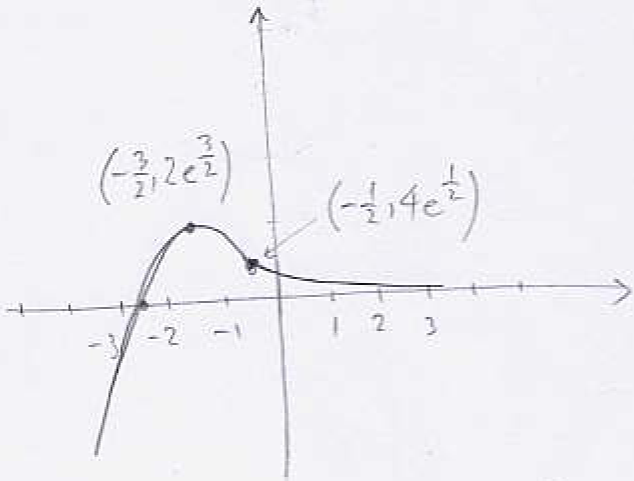
		$-\frac{1}{2}$	
$f''(x)$	−	0	+
$f(x)$	CD	IP	CU

c) $$\lim_{x\to\infty} \underbrace{e^{-x}}_{\to 0}\underbrace{(2x+5)}_{\to\infty} = \lim_{x\to\infty} \frac{\overset{\to\infty}{2x+5}}{\underset{\to\infty}{e^{x}}} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^{x}}$$
$$= \frac{2}{\infty} = 0$$

$$\lim_{x\to-\infty} e^{-x}(2x+5) = e^{\infty}\cdot(-\infty) = -\infty$$

d)

x	$f(x)$
$-\frac{3}{2}$	$e^{\frac{3}{2}}(2\cdot(-\frac{3}{2})+5) = 2e^{\frac{3}{2}}$
$-\frac{1}{2}$	$e^{\frac{1}{2}}(2\cdot(-\frac{1}{2})+5) = 4e^{\frac{1}{2}}$



9. (12pts) The body surface area A (in meters squared) of a person 169cm tall is given by the formula $A = \frac{13\sqrt{w}}{60}$, where w is the weight of the person in kilograms.

a) Find the body surface area of a person weighing 64kg.

b) Find the ROC of the body surface area with respect to weight when $w = 64$ (units?).

c) Use b) estimate the change in body surface area if weight changes by 3kg.

d) Use c) to estimate the body surface area of a person weighing 67kg .

a) $A(64) = \frac{13}{60} \cdot \sqrt{64} = \frac{13}{60} \cdot 8 = \frac{13 \cdot 2}{15} = \frac{26}{15}$

b) $A' = \frac{13}{60} \frac{1}{2\sqrt{w}} = \frac{13}{120\sqrt{w}}$

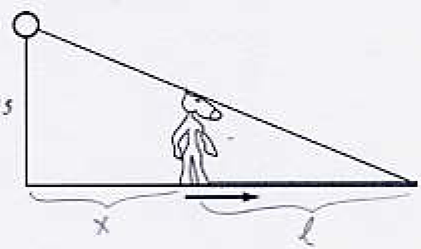
$A'(64) = \frac{13}{120 \cdot \sqrt{64}} = \frac{13}{960}$ m^2/kg

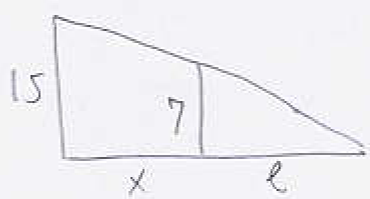
c) $\Delta A \approx A'(64) \cdot \Delta w = \frac{13}{960} \cdot 3 = \frac{13}{320}$ kg.

$$\begin{array}{r} 64 \cdot 26 \\ \hline 128 \\ 384 \\ \hline 1664 \end{array}$$

d) $A(67) \approx A(64) + \Delta A = \frac{26}{15} + \frac{13}{\underset{32 \cdot 2 \cdot 5}{320}} = \frac{64 \cdot 26 + 13 \cdot 3}{960} = \frac{1703}{960}$

10. (14pts) Somewhere on campus, seven-foot tall Dunker is walking away from a lamppost 15ft tall at a speed of 4 ft/second. How fast is length of Dunker's shadow on the ground changing when Dunker is 30ft away from the lamppost? What are the units?





Know: $\frac{dx}{dt} = 4$ ft/s

Need: $\frac{d\ell}{dt} = ?$

$\frac{\ell}{7} = \frac{\ell + x}{15}$

$15\ell = 7(\ell + x)$

$8\ell = 7x$

$\ell = \frac{7}{8}x \quad | \, d/dt$

$\ell' = \frac{7}{8}x'$

At any moment (not just when $x = 30$)

$\ell' = \frac{7}{8} \cdot 4 = \frac{7}{2}$ ft/s

$1+3\cdot\frac{2}{1}+4$

11. (12pts) Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + 3\frac{y}{x} + y^2 = 11$ at the point $(1, 2)$.

$x^2 + 3\cdot\frac{y}{x} + y^2 = 11 \quad \Big| \frac{d}{dx}$

$2x + 3\cdot\frac{y'\cdot x - y\cdot 1}{x^2} + 2yy' = 0 \quad \Big| \cdot x^2$

$2x^3 + 3(xy' - y) + 2x^2yy' = 0$

$y'(3x + 2x^2y) = 3y - 2x^3$

$y' = \frac{3y - 2x^3}{3x + 2x^2y}$

When $(x,y) = (1,2)$

$y' = \frac{3\cdot 2 - 2}{3 + 2\cdot 3} = \frac{4}{9}$

Equation of tangent line,

$y - 2 = \frac{4}{9}(x-1)$

$y = \frac{4}{9}x - \frac{4}{9} + 2$

$= \frac{4}{9}x + \frac{14}{9}$

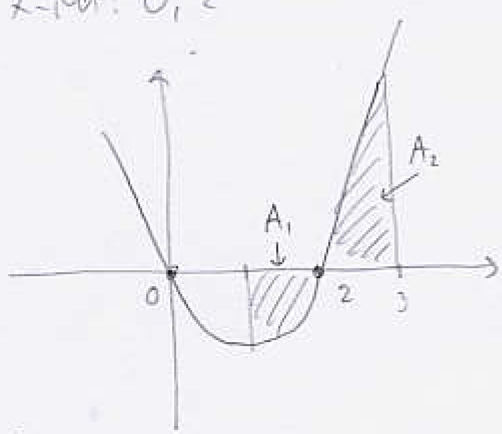
12. (12pts) The integral $\int_1^3 x^2 - 2x\,dx$ is given.
a) Draw a graph of $y = x^2 - 2x$ and state what the integral represents graphically. Can you tell from the picture if the integral is positive or negative?
b) Evaluate the integral and confirm your answer from a).

a) $y = x^2 - 2x$ is a parabola

$x^2 - 2x = 0$

$x(x-2) = 0$

x-int: $0, 2$



$\int_1^3 x^2 - 2x\,dx = -A_1 + A_2$

It appears $A_2 - A_1$ is positive,
since A_2 is bigger.

b) $\int_1^3 x^2 - 2x\,dx = \left(\frac{x^3}{3} - x^2\right)\Big|_1^3$

$= \frac{1}{3}(3^3 - 1) - (3^2 - 1) = \frac{26}{3} - 8$

$= \frac{2}{3}$, which is positive.

13. (10pts) Use the substitution rule to find the integral:

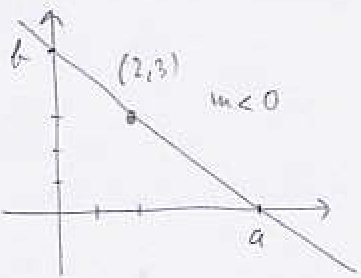
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{\cos^3 x}\,dx = \left[\begin{array}{ll} u=\cos x & x=\frac{\pi}{3},\ u=\frac{1}{2} \\ du=-\sin x\,dx & x=\frac{\pi}{4},\ u=\frac{\sqrt{2}}{2} \\ -du=\sin x\,dx & \end{array}\right] = \int_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} \frac{1}{u^3}(-du)$$

$\frac{1}{u^3}$

u^{-3}

$$= -\frac{u^{-2}}{-2}\Big|_{\frac{\sqrt{2}}{2}}^{\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{u^2}\Big|_{\sqrt{2}/2}^{1/2} = \frac{1}{2}\left(\frac{1}{(\frac{1}{2})^2}-\frac{1}{(\frac{\sqrt{2}}{2})^2}\right) = \frac{1}{2}(4-2) = 1$$

Bonus. (15pts) Draw a line of negative slope through the point $(2,3)$. Along with the axes, this line forms a right triangle in the first quadrant. Among all triangles obtained in this way, find the one with the smallest area.



$y-3=m(x-2)$

Find x- and y-intercepts a and b

$x=0,\quad y-3=-2m$

$y=3-2m=b$

$y=0\quad -3=m(x-2)$

$-\frac{3}{m}=x-2$

$x=2-\frac{3}{m}=a$

$$\text{Area}=\frac{1}{2}a\cdot b=\frac{1}{2}\left(2-\frac{3}{m}\right)(3-2m)$$

$$=\frac{1}{2}\left(6-\frac{9}{m}-4m+6\right)=\frac{1}{2}\left(12-4m-\frac{9}{m}\right)$$

Job: minimize $A(m)=\frac{1}{2}\left(12-4m-\frac{9}{m}\right)$ on $(-\infty,0)$

$$A'(m)=\frac{1}{2}\left(-4-9(-m^{-2})\right)=\frac{1}{2}\left(\frac{9}{m^2}-4\right)$$

$A'(m)=0$ when $\frac{9}{m^2}-4=0$

$\frac{9}{m^2}=4\qquad m^2=\frac{9}{4},\quad m=\pm\frac{3}{2}$

not in interval

Check if it is a min:

$$A''(m)=\frac{1}{2}\left(9(-2)m^{-3}\right)=-\frac{9}{m^3}$$

$A''\left(-\frac{3}{2}\right)=-\frac{9}{(-\frac{3}{2})^3}>0$ so A has a local max. at $m=-\frac{3}{2}$

Since there is only one critical point, A has an absolute min there. Minimal area is $A(-\frac{3}{2})$

$$=\frac{1}{2}\left(12-4(-\tfrac{3}{2})-9\cdot(-\tfrac{2}{3})\right)=\frac{1}{2}(12+6+6)=6$$