Spring '10/MAT 250/Exam 7 **Name:** *Show all your work.*

Find the following indefinite integrals.

1. (4pts) $\int 4x^2 - 3\sqrt{x}\,dx =$ $\int 4x^2 - 3x^{\frac{1}{2}}\,dx = 4\frac{x^3}{3} - 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{4}{3}x^3 + 2x^{\frac{3}{2}} + C$

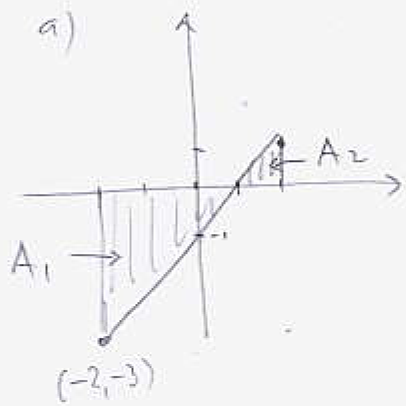
2. (4pts) $\int e^{3x+1}\,dx =$ $\frac{e^{3x+1}}{3} + C$

3. (4pts) $\int \sec(4x)\tan(4x)\,dx =$ $\frac{\sec(4x)}{4} + C$

$(\sec x)' = \sec x \tan x$

4. (15pts) Find $\int_{-2}^{2} x - 1\,dx$ in two ways (they'd better give you the same answer!):
a) Using the "area" interpretation of the integral. Draw a picture.
b) Using the Fundamental Theorem of Calculus.

a)



$\int_{-2}^{2} x-1\,dx = -A_1 + A_2 =$

$= -\frac{1}{2}3 \cdot 3 + \frac{1}{2}1 \cdot 1$

$= \frac{-9+1}{2} = -4$

b) $\int_{-2}^{2} x-1\,dx = \left(\frac{x^2}{2} - x\right)\Big|_{-2}^{2} = \frac{1}{2}\underbrace{(2^2 - (-2)^2)}_{0} - (2-(-2)) = 4$

5. (6pts) Write in sigma notation.

$$\frac{2}{3}+\frac{4}{9}+\frac{6}{27}+\frac{8}{81}+\frac{10}{243}= \sum_{i=1}^{5}\frac{2i}{3^i}$$

$3^1 \quad 3^2 \quad 3^3 \quad 3^4 \quad 3^5$

6. (10pts) Evaluate the definite integral:

$$\int_1^4 \frac{x^2-1}{\sqrt{x}}\,dx = \int_1^4 \frac{x^2-1}{x^{1/2}}\,dx = \int_1^4 x^{\frac{3}{2}} - x^{-\frac{1}{2}}\,dx$$

$$= \left(\frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}}\right)\Big|_1^4 = \frac{2}{5}\left(4^{\frac{5}{2}}-1^{\frac{5}{2}}\right)-2\left(4^{\frac{1}{2}}-1^{\frac{1}{2}}\right)$$

$$= \frac{2}{5}(32-1)-2(2-1) = \frac{62}{5}-2 = \frac{52}{5}$$

Use the substitution rule in the following integrals:

7. (9pts) $\int (6x^2-4x)(x^3-x^2+2)^5\,dx = \begin{bmatrix} u = x^3-x^2+2 \\ du = (3x^2-2x)dx \end{bmatrix} = \int 2u^5\,du$

$2(3x^2-2x)$

$$= \frac{2u^6}{6} = \frac{(x^3-x^2+2)^6}{3} + C$$

8. (10pts) $\int_{\cancel{0}\ \frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sec^2 x}{\tan^7 x}\,dx = \begin{bmatrix} u = \tan x & x=\frac{\pi}{4},\ u=\tan\frac{\pi}{4}=1 \\ du = \sec^2 x\,dx & x=\frac{\pi}{6},\ u=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}} \end{bmatrix}$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{du}{u^7} = \int_{\frac{1}{\sqrt{3}}}^{1} u^{-7}\,du = \frac{u^{-6}}{-6}\Big|_{\frac{1}{\sqrt{3}}}^{1} = -\frac{1}{6}\left(\frac{1}{1^6}-\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^6}\right)$$

$$= -\frac{1}{6}(1-27) =$$

$$= \frac{26}{6} = \frac{13}{3}$$

9. (8pts) Find the function f if $f'(x) = \frac{5}{x^4} - \frac{3}{x}$, and $f(1) = 7$.

$$f'(x) = 5x^{-4} - 3\cdot\frac{1}{x}$$

$$f(x) = 5\frac{x^{-3}}{-3} - 3\ln|x| + C$$

$$7 = f(1) = -\frac{5}{3} - 0 + C$$

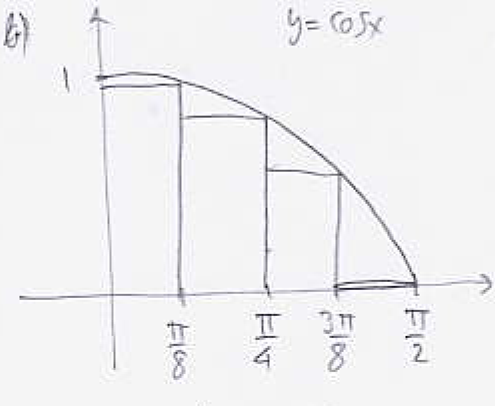
$$7 + \frac{5}{3} = C$$

$$C = \frac{26}{3}$$

$$f(x) = -\frac{5}{3x^3} - 3\ln|x| + \frac{26}{3}$$

10. (22pts) The function $f(x) = \cos x$, $0 \le x \le \frac{\pi}{2}$ is given.
a) Write down the expression that is used to compute R_4. Then compute R_4.
b) Illustrate with a diagram, where appropriate rectangles are clearly visible. What does R_4 represent?
c) Is R_4 an overestimate or an underestimate of the area?
d) Using the Fundamental Theorem of Calculus, evaluate $\int_0^{\frac{\pi}{2}} \cos x \, dx$. What is the error of R_4?

b)



$$\Delta x = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}$$

R_4 is the sum of areas of the shown rectangles

a) $R_4 = \frac{\pi}{8}\cos\frac{\pi}{8} + \frac{\pi}{8}\cos\frac{\pi}{4} + \frac{\pi}{8}\cos\frac{3\pi}{8} + \frac{\pi}{8}\cos\frac{\pi}{2}$

≈ 0.7908

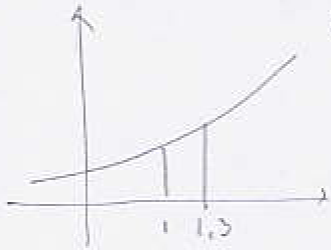
c) R_4 is an underestimate of the area.
(all rectangles are under the graph)

d) $\int_0^{\frac{\pi}{2}} \cos x\,dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0 = 1$

Error $= 1 - 0.7908 = 0.2092$

11. (8pts) Show that $0.82 \le \int_1^{1.3} e^{x^2}\,dx \le 1.63$. (Note: the antiderivative of e^{x^2} cannot be found among elementary functions, so don't try to do it by evaluating the integral.)



e^{x^2} is increasing

$$e^{1^2} \le e^{x^2} \le e^{1.3^2}$$

$$e(1.3-1) \le \int_1^{1.3} e^{x^2} \le e^{1.3^2}\cdot(1.3-1)$$

$$0.82 \le \int_1^{1.3} e^{x^2} \le 1.63$$

Bonus. (10pts) The gist of section 5.5 is this:
$\int_a^b$ rate of change of F = change of F from a to b. In other words, $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Use the fact above to solve the following problem. Water flows in and out of a tank at rate $3 - \frac{1}{2}t$ liters/minute. There were 5 liters of water in the tank at time $t = 0$.
a) By how much does the amount of water in tank change from $t = 0$ to $t = 4$?
b) How much water is in the tank at time $t = 4$?
c) By how much does the amount of water in tank change from $t = 4$ to $t = 10$?
d) How much water is in the tank at time $t = 10$?

$$V' = 3 - \tfrac{1}{2}t$$

a) $\Delta V = \int_0^4 3 - \frac{1}{2}t\,dt = 3(4-0) - \frac{1}{2}\int_0^4 t\,dt = 12 - \left(\frac{1}{2}\frac{t^2}{2}\Big|_0^4\right) = 12 - \frac{1}{4}(4^2 - 0) = 8\,\ell$

b) $V(4) = V(0) + \Delta V = 5 + 8 = 13$

c) $\Delta V = \int_4^{10} 3 - \frac{1}{2}t\,dt = 3(10-4) - \frac{1}{2}\int_4^{10} t\,dt = 18 - \left(\frac{1}{2}\frac{t^2}{2}\Big|_4^{10}\right) = 18 - \frac{1}{4}(10^2 - 4^2) = 18 - \frac{84}{4} = -3\,\ell$

(water level <u>decreased</u> from $t = 4$ to $t = 10$)

d) $V(10) = V(4) + \Delta V = 13 + (-3) = 10\,\ell$,