Spring '10/MAT 250/Exam 6 **Name:** Solution *Show all your work.*

Find the limits. You may use L'Hopital's rule.

1. (10pts) $$\lim_{x\to\infty} \frac{\overset{\to\infty}{x^2+3x-7}}{\underset{\to\infty}{e^x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{\overset{\to\infty}{2x+3}}{\underset{\to\infty}{e^x}} \overset{L'H}{=} \lim_{x\to\infty}\frac{2}{e^x} = \frac{2}{\infty} = 0$$

Which grows faster, x^2+3x-7 or e^x? Since the limit of the ratio $\frac{x^2+3x-7}{e^x}$ is 0, e^x grows faster.

2. (8pts) $$\lim_{x\to0}\frac{\overset{1-1=0}{\cos x-1}}{\underset{0}{x^2}} \overset{L'H}{=} \lim_{x\to0}\frac{\overset{\to0}{-\sin x}}{\underset{\to0}{2x}} = \lim_{x\to0}\frac{-\cos x}{2} = -\frac{1}{2}$$

3. (10pts) $$\lim_{x\to0+}\sqrt[5]{x}\ln x = \lim_{x\to0+}\frac{\overset{\to\infty}{\ln x}}{\underset{\to\infty}{x^{-1/5}}} \overset{L'H}{=} \lim_{x\to0+}\frac{\frac{1}{x}}{-\frac{1}{5}x^{-6/5}} = \lim_{x\to0+} -5x^{\overset{\overset{-1-(-\frac{6}{5})}{\downarrow}}{1/5}} = 0$$

4. (10pts) $$\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$$

$y = x^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln x$

$$\lim_{x\to\infty}\frac{\overset{\to\infty}{\ln x}}{\underset{\to\infty}{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = \frac{1}{\infty} = 0$$

5. (32pts) Let $f(x) = \ln(x^2+9)$. Draw an accurate graph of f by following the guidelines.

a) Find the intervals of increase and decrease, and local extremes.

b) Find the intervals of concavity and points of inflection.

c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

d) Use information from a)–d) to sketch the graph

$$f'(x) = \frac{1}{x^2+9}\cdot 2x = \frac{2x}{x^2+9} = 2\cdot\frac{x}{x^2+9}$$

$$f''(x) = 2\frac{1\cdot x^2+9 - x\cdot 2x}{(x^2+9)^2}$$

$$= 2\,\frac{9-x^2}{(x^2+9)^2} = \frac{2(9-x^2)}{(x^2+9)^2}$$

a) Critical points:

$f'(x)=0$ $\quad 2x=0,\ x=0$

$f'(x)$ not def. $\quad x^2+9=0$

$x^2=-9$ no sol,

$\frac{2x}{x^2+9}$ ← always positive

		0	
f'	−	0	+
f	↘	loc. min	↗

b) $f''(x)=0$ $\qquad f''(x)$ not def.

$9-x^2=0$ $\qquad x^2+9=0$, no sol.

$x^2=9$

$x=\pm3$ $\qquad f''(x) = \frac{2(9-x^2)}{(x^2+9)^2}$ ← graph is

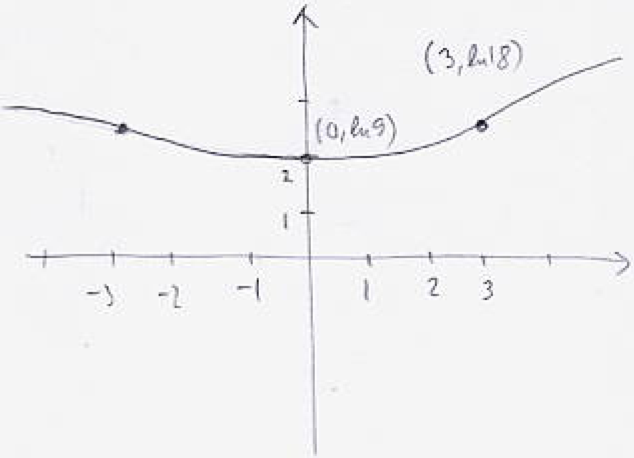
← always pos.

		-3		3	
f'	+	0	−	0	+
f	CU	IP	CD	IP	CU

x	f(x)
0	$\ln 9 \approx 2$
−3	$\ln 18 \approx 2.5$
3	$\ln 18 \approx 2.5$

c) $\lim_{x\to\infty} \ln(\underbrace{x^2+9}_{\to\infty}) = \ln\infty = \infty$

Same for $x\to-\infty$

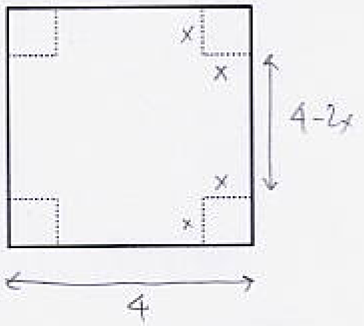


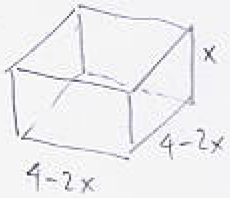
6. (8pts) Find the limit without using L'Hopital's rule.

$$\lim_{x\to\infty}\frac{5x^2-3x+1}{2x^2-x+2} = \lim_{x\to\infty}\frac{\cancel{x^2}\left(5-\frac{3}{x}+\frac{1}{x^2}\right)}{\cancel{x^2}\left(2-\frac{1}{x}+\frac{2}{x^2}\right)} = \frac{5-0+0}{2-0+0} = \frac{5}{2}$$

$\to 0 \quad \to 0$

7. (22pts) A square piece of cardboard has side length 4ft. Four equal-size square pieces are cut from the corners to produce a cross-like shape, whose "flaps" are lifted up to form a box without a top. Find the size of the cutout that produces the box with the largest volume.





$V = (4-2x)^2 \cdot x$

Must have $4-2x \geq 0$

$2x \leq 4$

$x \leq 2$

Job: maximize $V(x) = (2x-4)^2 \cdot x$ on $[0,2]$

$V'(x) = 2(2x-4)\cdot 2\cdot x + (2x-4)^2 = (2x-4)(4x+2x-4)$

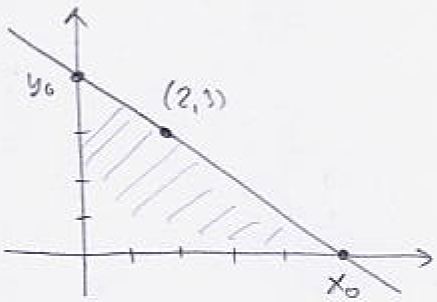
$= (2x-4)(6x-4)$

$V'(x)=0$ when $x=2$

or $x = \frac{4}{6} = \frac{2}{3}$

x	V(x)
0	0
2	0
$\frac{2}{3}$	$\left(2\cdot\frac{2}{3}-4\right)^2\frac{2}{3} = \left(-\frac{8}{3}\right)^2\cdot\frac{2}{3} = \frac{128}{27}$ ft^3

Bonus. (10pts) Draw a line of negative slope through the point (2, 3). Along with the axes, this line forms a right triangle in the first quadrant. Among all triangles obtained in this way, find the one with the smallest area.



Find intercepts of a line of slope $m<0$

Eq. of line:

$y-3=m(x-2)$

Set $y=0$

$-3=m(x-2)$

$-\frac{3}{m}=x-2$

$x=2-\frac{3}{m}$

$x_0=2-\frac{3}{m}$

Set $x=0$

$y-3=-2m$

$y=-2m+3$

$y_0=-2m+3$

$$A=\frac{1}{2}x_0y_0=\frac{1}{2}\left(2-\frac{3}{m}\right)(-2m+3)=\frac{1}{2}\left(-4m+6+6-\frac{9}{m}\right)=\frac{1}{2}\left(12-4m-\frac{9}{m}\right)$$

Job: minimize $A(m)=\frac{1}{2}\left(12-4m-\frac{9}{m}\right)$ on $(-\infty,0)$

$A'=\frac{1}{2}\left(-4+9m^{-2}\right)$

$A'=0 \quad -4+\frac{9}{m^2}=0$

$\frac{9}{m^2}=4$

$m^2=\frac{9}{4} \quad m=\pm\frac{3}{2}$

not in interval.

$A''=\frac{1}{2}\left(-18m^{-3}\right)=-\frac{9}{m^3}$

$A''\left(-\frac{3}{2}\right)=-\frac{9}{\left(-\frac{3}{2}\right)^3}>0$ ✓

So it is a local min.

Since it is the only critical point, it must also be an absolute min.

$$A\left(-\frac{3}{2}\right)=\frac{1}{2}\left(12-4\cdot\left(-\frac{3}{2}\right)-9\cdot\left(-\frac{2}{3}\right)\right)=\frac{1}{2}(12+6+6)=12$$