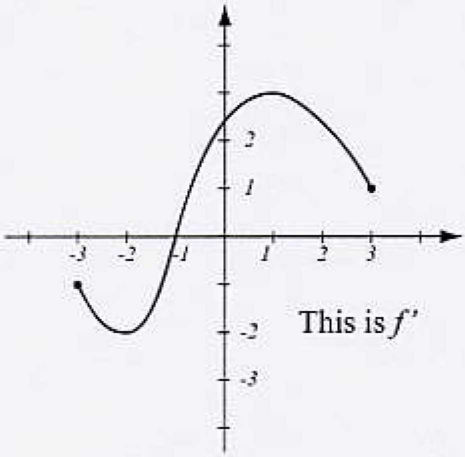
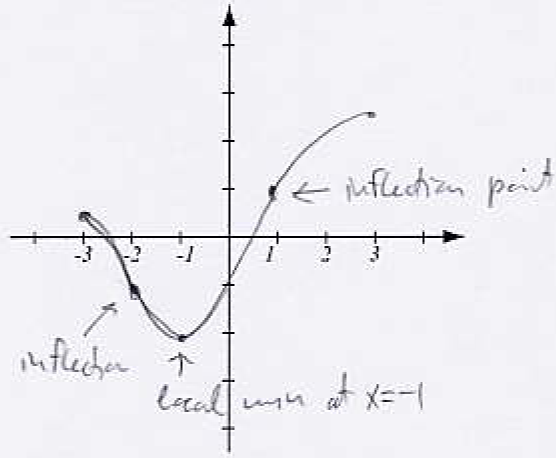
Spring '10/MAT 250/Exam 5 **Name:** Solution *Show all your work.*

1. (~~15~~ 19pts) Let the domain of f be $[-3,3]$. The graph of its derivative f' is drawn below. Use the graph to answer:
a) What are the intervals of increase and decrease of f?
b) What are the intervals of concavity of f?
c) Use the information gathered in a) and b) to draw one possible graph of f.





a) f $\begin{matrix}\text{incr}\\ \text{decr}\end{matrix} \iff f' \begin{matrix} >0 \\ <0 \end{matrix}$

f incr. on $(-1,3)$
f decr. on $(-3,-1)$

b) f $\begin{matrix}\text{conc. up}\\ \text{c \quad dn}\end{matrix} \iff f'$ $\begin{matrix}\text{incr}\\ \text{decr.}\end{matrix}$

f conc. up on $(-2,1)$
f conc. down on $(-3,-2)\cup(1,3)$

2. (10pts) A rubber ball is being inflated. Using linear approximation, estimate by how much volume changes if radius changes from $r = 6$in to $r = 6.5$in. (The volume of a ball of radius r is $V = \frac{4\pi}{3}r^3$.)

$\Delta V \approx V'(6)\cdot \Delta r$

$V' = \frac{4\pi}{3}\cdot 3r^2 = 4\pi r^2$

$\Delta V \approx 4\pi \cdot 6^2 \cdot \underbrace{0.5}_{\Delta r}$

$= 4\pi \cdot 36 \cdot \frac{1}{2}$

$= 72\pi \text{ in}^3$

3. (1~~2~~3pts) Verify the Mean Value Theorem for the function $f(x) = \sqrt{x}$ on the interval $[1, 4]$.

f is cont. on $[1,4]$
diff. on $(1,4)$

There exists a c in $(1,4)$ such that $f'(c) = \dfrac{f(4)-f(1)}{4-1}$

$= \dfrac{\sqrt{4}-\sqrt{1}}{4-1} = \dfrac{1}{3}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$\dfrac{1}{2\sqrt{x}} = \dfrac{1}{3}$

$\dfrac{1}{\sqrt{x}} = \dfrac{2}{3}$

$\sqrt{x} = \dfrac{3}{2}$

$x = \dfrac{9}{4} = 2\frac{1}{4}$, which is indeed in $(1,4)$

4. (~~20~~ 21pts) Let $f(x) = \dfrac{x+3}{x^2+7}$.

a) Find the cricital points and the intervals of increase and decrease of f. Determine where f has a local maximum or minimum.

b) Using information found in a), draw a rough sketch of f.

c) <u>D.T.</u> Initial here if you are glad you were not asked to find the second derivative of f. (But if you really want to, see the bonus).

a) $f'(x) = \dfrac{1\cdot(x^2+7) - (x+3)\cdot 2x}{(x^2+7)^2}$

$= \dfrac{x^2+7-2x^2-6x}{(x^2+7)^2}$

$= \dfrac{-x^2-6x+7}{(x^2+7)^2}$

Crit. pts: $f'(x) = 0$

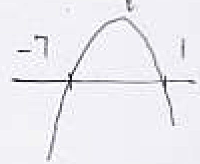
$-x^2-6x+7=0$

$x^2+6x-7=0$

$(x+7)(x-1)=0$

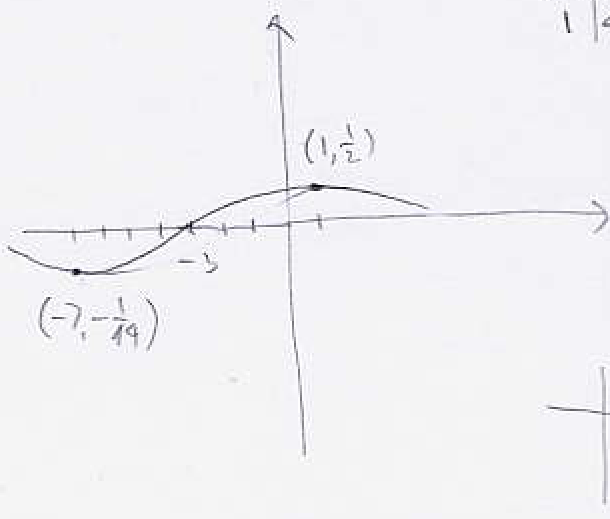
$x=-7, 1$

Since $(x^2+7)^2$ is always positive, sign of f' only depends on $-x^2+6x+7$



		-7		1	
$-x^2+6x+7$	$-$	0	$+$	0	$-$
f	↘	loc min	↗	loc. max	↘

x	$f(x)$
-7	$-4/56 = -\frac{1}{14}$
1	$4/8 = 1/2$



$\sin^2\theta - \cos\theta$

5. (16pts) Let $f(\theta) = \cancel{\cos^2\theta +} \sin\theta$. Find the absolute minimum and maximum values of f on the interval $[0, \pi]$.

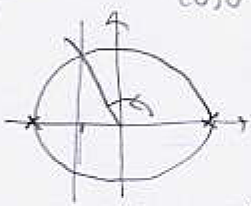
$f'(\theta) = 2\sin\theta\cos\theta - (-\sin\theta)$

$2\sin\theta\cos\theta + \sin\theta = 0$

$\sin\theta(2\cos\theta + 1) = 0$

$\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = 0, \pi$ $\qquad \cos\theta = -\frac{1}{2}$



$\theta = \frac{2\pi}{3}$

Evaluate:

θ	$\sin^2\theta - \cos\theta$	
$\frac{2\pi}{3}$	$\left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$	max
0	$0 - 1 = -1$	min
π	$0 - (-1) = 1$	

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6. (20pts) Let $f(x) = e^{-x}(2x + 3)$.
a) Find the intervals of concavity and points of inflection for f.
b) Find the critical points and use a) to determine which are local minima or maxima.

a) $f'(x) = e^{-x}(-1)(2x+3) + e^{-x} \cdot 2$

$= e^{-x}(-2x - 3 + 2)$

$= e^{-x}(-2x - 1)$

$f''(x) = e^{-x}(-1)(-2x-1) + e^{-x}(-2)$

$= e^{-x}(2x + 1 - 2)$

$= e^{-x}(2x - 1)$

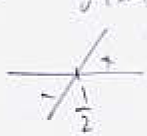
$f''(x) = 0 \qquad e^{-x}(2x-1) = 0$

$\uparrow$ always $> 0 \qquad 2x - 1 = 0$

$x = \frac{1}{2}$

		$\frac{1}{2}$	
f''	$-$	0	$+$
f	conc down	$\uparrow$ inflection pt	conc up

$y = 2x - 1$



b) $f'(x) = 0$

$e^{-x}(-2x - 1) = 0$

$\uparrow$ $>0 \qquad -2x - 1 = 0$

$x = -\frac{1}{2}$

Since $f''(-\frac{1}{2}) < 0$ $\frown$

f has a local max. at $x = -\frac{1}{2}$

Bonus

7. ~~(8pts)~~ After two hours of driving, Frank was 70 miles away from his starting point. After three hours he was 120 miles from the starting point. There are three values that you can be sure his speedometer displayed during his drive. What are they? Justify your answer with a theorem. (Don't say one of the values is zero, because he might not have been at a standstill when we started observing.)

$s(t)=$ position

$s(0)=0$

$s(2)=70$

$s(3)=120$

interval	Average velocity on interval
$[0,2]$	$\frac{70-0}{2-0}=35$
$[2,3]$	$\frac{120-70}{3-2}=50$
$[0,3]$	$\frac{120-0}{3-0}=40$

By MVT, on every interval, there is a time so that instantaneous velocity equals average velocity. Therefore, at some time the velocity was 35mph, 50mph and 40mph.

Bonus. (10pts) Find the intervals of concavity for the function in problem 4.

$$f'(x)=\frac{-x^2-6x+7}{(x^2+7)^2}$$

$$f''(x)=\frac{(-2x-6)(x^2+7)^2-(-x^2-6x+7)\cdot 2(x^2+7)\cdot 2x}{(x^2+7)^4}$$

$$=\frac{(x^2+7)\left((-2x-6)(x^2+7)+(x^2+6x-7)4x\right)}{(x^2+7)^4}$$

$$=\frac{-2x^3-6x^2-14x-42+4x^3+24x^2-28x}{(x^2+7)^3}$$

$$=\frac{2x^3+18x^2-42x-42}{(x^2+7)^3}$$

$$=\frac{2(x^3+9x^2-21x-21)}{(x^2+7)^3}$$

$27+81-63-21$

$-1+9+21-21$

$f''(x)=0$ is hard to solve since it involves a cubic equation, so we will leave it at that.