Spring '10/MAT 250/Exam ~~3~~4 Name: Solution *Show all your work.*

Differentiate and simplify where appropriate:

1. (5pts) $\frac{d}{dx}\ln(x^2-3x+1) = \frac{1}{x^2-3x+1}\cdot(2x-3) = \frac{2x-3}{x^2-3x+1}$

2. (5pts) $\frac{d}{d\theta}5^{\sin\theta} = \ln 5\cdot 5^{\sin\theta}\cdot\cos\theta$

3. (6pts) $\frac{d}{dx}x^2\arcsin x = 2x\arcsin x + x^2\cdot\frac{1}{\sqrt{1-x^2}} = 2x\arcsin x + \frac{x^2}{\sqrt{1-x^2}}$

4. (9pts) $\frac{d}{dx}\ln\frac{(x+1)^2}{(3x-2)^4} = \frac{d}{dx}\left(2\ln(x+1) - 4\ln(3x-2)\right)$

$= 2\frac{1}{x+1} - 4\cdot\frac{1}{3x-2}\cdot 3 = \frac{2}{x+1} - \frac{12}{3x-2} = \frac{2(3x-2)-12(x+1)}{(x+1)(3x-2)}$

$= \frac{6x-4-(12x+12)}{\quad} = \frac{-6x-16}{(x+1)(3x-2)} = -\frac{2(3x+8)}{(x+1)(3x-2)}$

5. (9pts) $\frac{d}{dt}\arctan\left(\frac{t^2-1}{t}\right) = \frac{1}{1+\left(\frac{t^2-1}{t}\right)^2}\cdot\frac{2t\cdot t-(t^2-1)\cdot 1}{t^2} = \frac{1}{1+\frac{t^4-2t^2+1}{t^2}}\cdot\frac{t^2+1}{t^2}$

$= \frac{t^2+1}{t^2+t^4-2t^2+1} = \frac{t^2+1}{t^4-t^2+1}$

6. (12pts) Use implicit differentiation to find the equation of the tangent line to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $(-2, \frac{3\sqrt{3}}{2})$. Draw a picture.

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \quad \Big| \frac{d}{dx}$$

$$\frac{2x}{16} + \frac{2yy'}{9} = 0$$

$$\frac{2yy'}{9} = -\frac{x}{8}$$

$$y' = -\frac{9x}{16y}$$

At $(-2, \frac{3\sqrt{3}}{2})$ it is

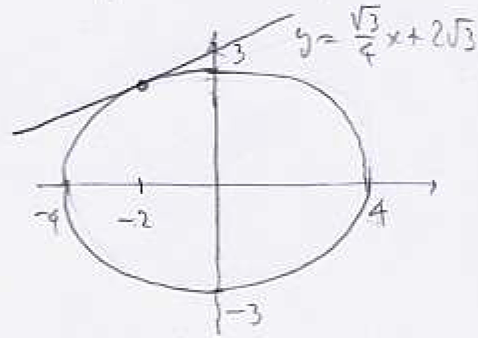
$$y' = -\frac{9\cdot(-2)}{16\cdot\frac{3\sqrt{3}}{2}} = \frac{18}{24\sqrt{3}}$$

$$= \frac{3}{4\sqrt{3}} = \frac{\sqrt{3}}{4}$$

Tan line:

$$y - \frac{3\sqrt{3}}{2} = \frac{\sqrt{3}}{4}(x+2)$$

$$y = \frac{\sqrt{3}}{4}x + \frac{4\sqrt{3}}{2} = \frac{\sqrt{3}}{4}x + 2\sqrt{3}$$



7. (10pts) Use implicit differentiation to find y'.

$$xe^y = \frac{x}{y} \quad \Big| \frac{d}{dx}$$

$$1e^y + xe^y y' = \frac{1\cdot y - xy'}{y^2} \quad \Big| \cdot y^2$$

$$y^2e^y + xy^2e^y y' = y - xy'$$

$$xy^2e^y y' + xy' = y - y^2e^y$$

$$x(y^2e^y + 1)y' = y(1-y)e^y$$

$$y' = \frac{y(1-y)e^y}{x(y^2e^y+1)}$$

8. (8pts) Use logarithmic differentiation to find the derivative of $y = x^{\cos x}$.

$$y = x^{\cos x} \quad \Big| \ln$$

$$\ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \cdot \ln x \quad \Big| \frac{d}{dx}$$

$$\frac{1}{y}\cdot y' = -\sin x \cdot \ln x + \cos x \cdot \frac{1}{x}$$

$$y' = y(\_\_\_\_) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right)$$

9. (10pts) Let $f(x) = x^3 + 2x^2 + 5x$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find $g'(8)$.

$$g'(8) = \frac{1}{f'(g(8))} = \frac{1}{f'(1)} = \frac{1}{3\cdot 1^2 + 4\cdot 1 + 5} = \frac{1}{12}$$

$f'(x) = 3x^2 + 4x + 5$

$g(8) = x$ where $f(x) = 8$

$x^3 + 2x^2 + 5x = 8$

$x = 1$

10. (10pts) Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal.

$3x^2 + 4y^2 + 3xy = 24 \quad \Big| \frac{d}{dx}$

$6x + 8yy' + 3(1\cdot y + xy') = 0$

$6x + 8yy' + 3y + 3xy' = 0$

Have to have $y' = 0$ so

$6x + 3y = 0$

$y = -\frac{6x}{3} = -2x$

put $y = -2x$ back in equation.

$3x^2 + 4(-2x)^2 + 3x(-2x) = 24$

$3x^2 + 16x^2 - 6x^2 = 24$

$13x^2 = 24$

$x^2 = \frac{24}{13}$ $\quad x = \pm\sqrt{\frac{24}{13}} = \frac{\pm 2\sqrt{6}}{13}$

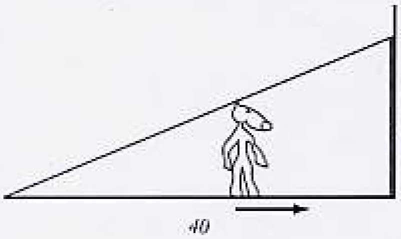
$y = \mp \frac{4\sqrt{6}}{13}$

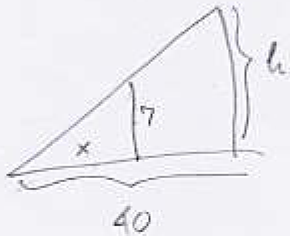
So points are:

$\left(\frac{2\sqrt{6}}{13}, -\frac{4\sqrt{6}}{13}\right)$

$\left(-\frac{2\sqrt{6}}{13}, \frac{4\sqrt{6}}{13}\right)$

11. (16pts) Somewhere on campus, a spotlight is shining on a wall 40ft away. Seven-foot tall Dunker is walking from the spotlight toward the wall at rate 4 ft/second. How fast is the height of Dunker's shadow on the wall changing when Dunker is 30ft away from the spotlight? What are the units?





Know: $x' = 4$ ft/s

Need h' when $x = 30$

From similar triangles:

$$7x^{-1} \to \quad \frac{7}{x} = \frac{h}{40} \quad \Big| \frac{d}{dt}$$

$$7\cdot(-1)x^{-2}x' = \frac{h'}{40}$$

$$h' = -\frac{280x'}{x^2}$$

When $x = 30$, get

$$h' = -\frac{280\cdot 4}{30^2} = -\frac{\overset{56}{\cancel{280\cdot 4}}}{9\cdot\underset{\underset{5}{\cancel{25}}}{\cancel{100}}} = -\frac{56}{45} \text{ ft/sec}$$

negative because length of shadow is decreasing.

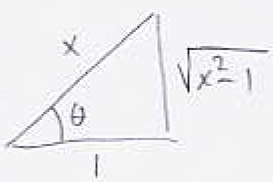
Bonus. (10pts) Let $f(\theta) = \sec\theta$, and let g be the inverse of f. Use the theorem on derivatives of inverses to find the general expression for $g'(x)$. (A triangle with θ and x will help you.)

$\sec\theta = x$

so $\theta = g(x)$

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(\theta)} = \frac{1}{\sec\theta\tan\theta} = \frac{1}{x\cdot\frac{\sqrt{x^2-1}}{x}}$$

$$g'(x) = \frac{1}{x\sqrt{x^2-1}}$$



$\frac{1}{\cos\theta} = x$

$\cos\theta = \frac{1}{x}$