Spring '10/MAT 250/Exam 2 **Name:** Solution *Show all your work.*

1. (16pts) Differentiate and simplify where appropriate:

$$\frac{d}{dx}\left(7x^4 - \frac{1}{x^8} + \sqrt[5]{x^{17}} + \sqrt{7}\right) = \frac{d}{dx}\left(7x^4 - x^{-8} + x^{\frac{17}{5}} + \sqrt{7}\right)$$ constant

$$= 28x^3 - (-8)x^{-9} + \frac{17}{5}x^{\frac{12}{5}} + 0$$

$$= 28x^3 + 8x^{-9} + \frac{17}{5}x^{\frac{12}{5}}$$

$$\frac{d}{dx}\frac{4x^2 + 5\sqrt[3]{x} + 1}{\sqrt{x}} = \frac{d}{dx}\left(\frac{4x^2 + 5x^{\frac{1}{3}} + 1}{x^{\frac{1}{2}}}\right) = \frac{d}{dx}\left(4x^{\frac{3}{2}} + 5x^{-\frac{1}{6}} + x^{-\frac{1}{2}}\right)$$

$$= 4\cdot\frac{3}{2}x^{\frac{1}{2}} + 5\left(-\frac{1}{6}\right)x^{-\frac{7}{6}} + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$= 6x^{\frac{1}{2}} - \frac{5}{6}x^{-\frac{7}{6}} - \frac{1}{2}x^{-\frac{3}{2}}$$

$$\frac{d}{dt}(Ae^t + B) = Ae^t$$

2. (14pts) Find the equation of the tangent line to the curve $y = x^2 - 2x - 8$ at the point $a = 0$. Sketch the curve and the tangent line.

$y' = 2x - 2$

$y'(0) = -2$

$y(0) = -8$

Equation of tan. line:

$y - (-8) = -2(x - 0)$

$y = -2x - 8$

Sketch:

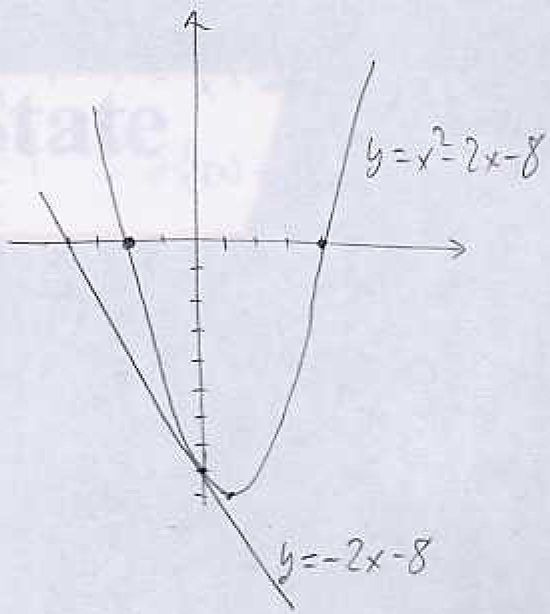
x-int:

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$4, -2$

vertex $= (1, -9)$



3. (22pts) Find the following limits algebraically.

$\lim_{x\to 7} \frac{x^2-2x-35}{x-7} = \lim_{x\to 7} \frac{\cancel{(x-7)}(x+5)}{\cancel{x-7}} = \lim_{x\to 7}(x+5)\ 12$

$\lim_{x\to 4} \frac{x-\cancel{4}\,9}{\sqrt{x}-3} = \lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9}\frac{\cancel{(x-9)}(\sqrt{x}+3)}{\cancel{x-9}} = \lim_{x\to 9}(\sqrt{x}+3) = 6$

$\lim_{x\to 0} \frac{\tan(5x)}{\sin(2x)} = \lim_{x\to 0} \frac{\frac{\sin(5x)}{\cos(5x)}}{\sin(2x)} = \lim_{x\to 0} \frac{\sin(5x)}{\sin(2x)\cdot\cos(5x)}\cdot\frac{2x}{2x}\cdot\frac{5x}{5x} =$

$= \lim_{x\to 0} \underbrace{\frac{\sin(5x)}{5x}}_{\to 1}\cdot\underbrace{\frac{2x}{\sin(2x)}}_{\to 1}\cdot\frac{5\cancel{x}}{2\cancel{x}\cos(5x)} = 1\cdot 1\cdot\frac{5}{2\underbrace{\cos 0}_{=1}} = \frac{5}{2}$

4. (10pts) Find $\lim_{x\to 0+}(e^x-1)\left(1+\cos\frac{1}{x}\right)$. Use the theorem that rhymes with what is halfway between your feet and hips.

$-1 \le \cos\frac{1}{x} \le 1 \quad \Big| +1$

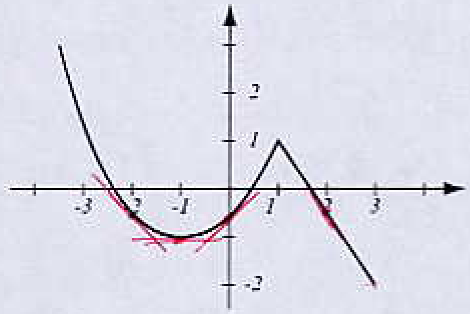
$0 \le 1+\cos\frac{1}{x} \le 2 \quad \Big| \cdot \underbrace{e^x-1}_{>0 \text{ for } x>0}$

$0\cdot(e^x-1) \le (e^x-1)\left(1+\cos\frac{1}{x}\right) \le 2(e^x-1)$

$\left.\begin{array}{l}\lim_{x\to 0+} 0 = 0 \\ \lim_{x\to 0+} 2(e^x-1) = 2(e^0-1) = 0\end{array}\right\}$ Limits are equal, so by the squeeze theorem, $\lim_{x\to 0+}(e^x-1)\left(1+\cos\frac{1}{x}\right) = 0$

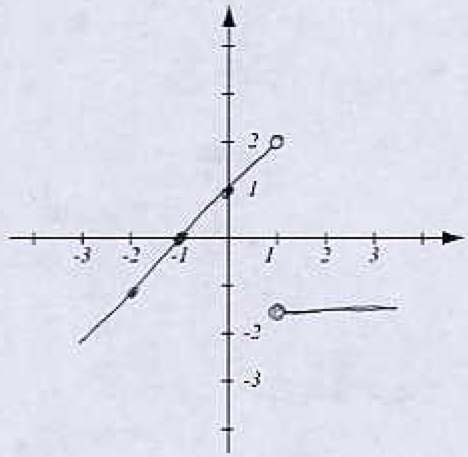
5. (16pts) The graph of the function $f(x)$ is shown at right.

a) Use the graph to fill out the table below with your estimates, noting if any of the derivatives do not exist.
b) In the second coordinate system, sketch the graph of the function $f'(x)$, with help from results in a).



x	-2	-1	0	1	2
$f'(x)$	-1	0	1	d.n.e	$-\frac{3}{2}$

sharp point



6. (12pts) Let $f(x) = 5x^2 + 2x - 1$.
a) Use the limit definition of the derivative to find the derivative of the function.
b) Check your answer by taking the derivative of f.

a) $$f'(a) = \lim_{x\to a} \frac{5x^2+2x-1-(5a^2+2a-1)}{x-a}$$

$$= \lim_{x\to a} \frac{5x^2+2x-5a^2-2a}{x-a}$$

$$= \lim_{x\to a} \frac{5(x^2-a^2)+2(x-a)}{x-a}$$

$$= \lim_{x\to a} \frac{5(x-a)(x+a)+2(x-a)}{x-a}$$

$$= \lim_{x\to a} \frac{(x-a)(5(x+a)+2)}{x-a} =$$

$$= \lim_{x\to a} (5(x+a)+2) = 10a+2$$

$$f'(x) = 10x+2$$

b) $f'(x) = 10x+2$

7. (10pts) Consider the limit below. It represents a derivative $f'(a)$.
a) Find f and a.
b) Use the infomation above to find the limit.

$$\lim_{x\to 2}\frac{x^3-8}{x-2} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \underset{\uparrow \atop b)}{=} \frac{d}{dx}x^3\Big|_{x=2} = 3x^2\Big|_{x=2} = 12$$

$a=2$

$f(x)=x^3$

$f(a)=2^3=8$

Bonus. (10pts) Let $m_b = \lim_{x\to 0}\frac{b^h-1}{h}$ and let $f(x)=b^x$, $b>0$, $b\neq 1$.
a) Use the limit definition of the derivative to show that $m_b = f'(0)$.
b) Use the limit definition of the derivative to show that $f'(x) = m_b\cdot b^x$.

a) $f'(0) = \lim_{h\to 0}\frac{b^{0+h}-b^0}{h} = \lim_{h\to 0}\frac{b^h-1}{h}$

b) $f'(x) = \lim_{h\to 0}\frac{b^{x+h}-b^x}{h} = \lim_{h\to 0}\frac{b^x(b^h-1)}{h} = b^x\cdot\underbrace{\lim_{h\to 0}\frac{b^h-1}{h}}_{m_b} = m_b\, b^x$

Using $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$x=0$ in a)

general x in b)