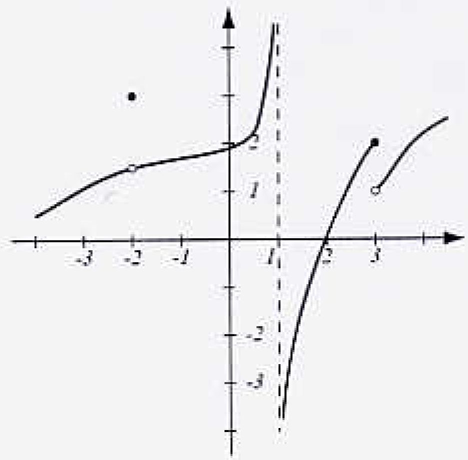
Spring '10/MAT 250/Exam 1 **Name:** Solution *Show all your work.*

1. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.



$\lim_{x\to 1^+} f(x) = -\infty$

$\lim_{x\to 3^-} f(x) = 2$

$\lim_{x\to 3^+} f(x) = 1$

$\lim_{x\to 3} f(x) =$ d.n.e one-sided limits are not equal

$\lim_{x\to -2} f(x) = 1.5$

$f(-2) = 3$

List points where f is not continuous and justify why it is not continuous at those points.

Not cont. at $x=-2$, where $\lim_{x\to -2} f(x) \neq f(-2)$

Not cont. at $x=3$, where $\lim_{x\to 3} f(x)$ does not exist.

Not cont. at $x=1$, where $f(1)$ is not defined

2. (4pts) Find the following limit algebraically (no need to justify, other than showing the computation).

$\lim_{x\to 2}(x^2+4x-6) = 2^2+4\cdot 2-6 = 4+8-6 = 6$

3. (8pts) Let $\lim_{x\to 3} f(x) = 4$ and $\lim_{x\to 3} g(x) = 7$. Use limit laws to find the limit below and show each step.

$$\lim_{x\to 3}\frac{3f(x)+7}{x^2-f(x)g(x)} = \frac{\lim_{x\to 3}(3f(x)+7)}{\lim_{x\to 3}(x^2-f(x)g(x))} = \frac{3\lim_{x\to 3}f(x)+\lim_{x\to 3}7}{\underbrace{\lim_{x\to 3}x^2}_{=\lim_{x\to 3}x\cdot\lim_{x\to 3}x}-\lim_{x\to 3}f(x)\cdot\lim_{x\to 3}g(x)}$$

$$= \frac{3\cdot 4+7}{3\cdot 3-4\cdot 7}$$

$$= \frac{19}{-19} = -1$$

4. (10pts) Find the domain of $f(x) = \dfrac{x^2}{\sqrt{e^x}}$. Then explain, using continuity laws, why the function is continuous on its domain.

Since $e^x > 0$ for all x,
$\dfrac{x^2}{\sqrt{e^x}}$ is defined for all x

Domain $= \mathbb{R}$.

x^2 : continuous, as a polynomial
e^x : continuous, as an exponential function
$\sqrt{\ }$: continuous, as inverse to x^2.

$\sqrt{e^x}$ is composite of cont. functions, hence continuous

$\dfrac{x^2}{\sqrt{e^x}}$ is a quotient of cont. functions, hence continuous.

5. (16pts) The height of a turnip t seconds after getting thrown upwards with initial velocity 20 meters per second is given by $h(t) = 20t - 5t^2$ (in meters).
a) Find the average velocities of the object over six short intervals of time, three of them beginning with 1.5, and three ending with 1.5. Show the table of values.
b) Use the information in a) to find the instantaneous velocity of the turnip at $t = 1.5$.

a) $h(1.5) = 20 \cdot 1.5 - 5 \cdot 1.5^2 = 18.75$

Average velocity on time interval $[1.5, t]$ is

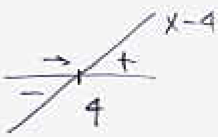
$$\frac{h(t) - h(1.5)}{t - 1.5} = \frac{20t - 5t^2 - 18.75}{t - 1.5}$$

interval	average velocity
[1.5, 1.51]	4.95
[1.5, 1.501]	4.995
[1.5, 1.5001]	4.9995
[1.49, 1.5]	5.05
[1.499, 1.5]	5.005
[1.499, 1.5]	5.0005

b) It appears that instantaneous velocity at time $t = 1.5$ is 5 m/s

6. (10pts) Find the following limit algebraically (do not use the calculator) and justify.

$$\lim_{x \to 4-} \frac{5x+3}{x-4} = \frac{23}{0-} = -\infty$$

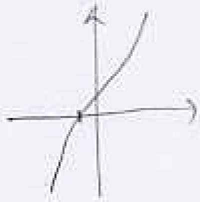


$\dfrac{23}{\text{small neg}}$ = large neg.

7. (18pts) The equation $e^x = x^2$ is given.
a) Use the Intermediate Value Theorem to show that this equation has at least one solution. Write a nice sentence that shows how you are using the IVT.
b) Use the bisection method and your calculator to find an interval of width less than 0.05 that contains your solution. Show every intermediate step.

a) $e^x = x^2$

$e^x - x^2 = 0$



Let $f(x) = e^x - x^2$. It is continuous on $\mathbb{R}$.

$f(0) = 1$

$f(-2) = -3.86$ $\quad -3.86 < 0 < 1$

Since $f(-2) < 0 < f(0)$, by Intermediate Value Theorem, there is a c in $(-2, 0)$ so that $f(c) = 0$.

b)

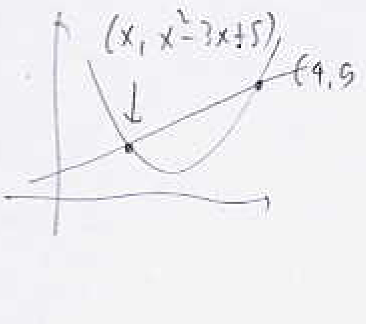
interval	midpt	f(midpt.)
$[-2, 0]$	-1	$-$
$[-1, 0]$	$-\frac{1}{2}$	$+$
$[-1, -0.5]$	-0.75	$-$
$[-0.75, -0.5]$	-0.625	$+$
$[-0.75, -0.625]$	-0.6875	$+$
$[-0.75, -0.6875]$	-0.71875	$-$

$[-0.71875, -0.6875]$ has width 0.03, so is the desired interval containing the solution.

8. (8pts) Let $f(x) = x^2 - 3x + 5$, and let $P = (4, 9)$. If $Q = (x, f(x))$ is another (general) point on the graph of f, write the formula for the slope of the secant line PQ and simplify.

$$\text{Slope} = \frac{f(x) - f(x_1)}{x - x_1} = \frac{x^2 - 3x + 5 - 9}{x - 4} = \frac{x^2 - 3x - 4}{x - 4} = \frac{(x-4)(x+1)}{x-4}$$

$$= x + 1$$



9. (10pts) Consider the limit $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$.

a) Use your calculator to find the limit correct to six decimal places — write down the table on paper. Make a guess as to what the limit is exactly.

b) What does the calculator give you if you take an x very close to 0? Does this alter your estimate of the limit? Why or why not?

x	$\dfrac{\sin x - x}{x^3}$
0.1	−0.166583
0.01	−0.166685
0.001	−0.166666
0.0001	−0.166666
−0.1	−0.166583
−0.01	−0.166658
−0.001	−0.166666
−0.0001	−0.166666
10^{-5}	−0.1666
10^{-6}	0
10^{-7}	0

a) It appears the limit is −0.166666 to six decimals, or $-\frac{1}{6}$

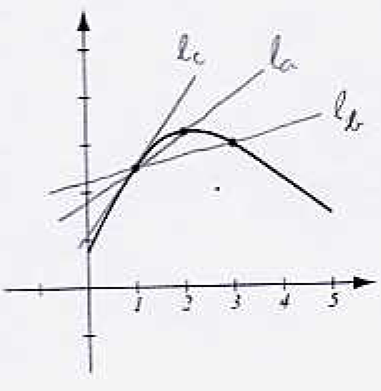
b) Starting with $x = 10^{-5}$ the calculator starts giving unexpected numbers. This is probably due to rounding error in calculating $\sin x$, so we will stay with our original estimate of $-\frac{1}{6}$.

Bonus. (10pts) Below is the graph of the outdoor temperature (in °C) measured t hours after noon. Use it to put the following three numbers in increasing order. No computation is needed, but justify your answer.

a = average rate of change from $t = 1$ to $t = 2$ – slope of secant line ℓ_a

b = average rate of change from $t = 1$ to $t = 3$ – slope of secant line ℓ_b

c = instantaneous rate of change at $t = 1$ → slope of tangent line ℓ_c.



From picture, it is clear that

$b < a < c$