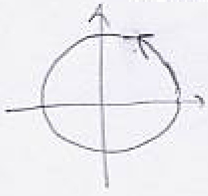
Fall '10/MAT 309/Worksheet 5 **Name:** Solution *Show all your work.*

For the following problems, you need to find the statements of the three major theorems in 17.1–17.3. Note the orientation conventions in those theorems.

1. (20pts) Verify Green's Theorem for the line integral $\oint_C xy\,dx + y\,dy$, where C is the unit circle, oriented Counter clockwise (see section 17.1).



$x = \cos t$
$y = \sin t$
$t \in [0, 2\pi]$

$$\int_C xy\,dx + y\,dy = \int_0^{2\pi} \langle \cos t \sin t, \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} -\cos t \sin^2 t + \sin t \cos t\,dt = \int_0^{2\pi} (\sin t - \sin^2 t)\cos t\,dt$$

$$= \begin{bmatrix} u = \sin t & t = 2\pi,\ u = 0 \\ du = \cos t\,dt & t = 0,\ u = 0 \end{bmatrix} = \int_0^0 u - u^2\,du = 0$$

$Q = y$
$P = xy$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 0 - x\,dA = -\iint_D x\,dA = 0$$

are equal

by symmetry, integration over unit circle

Green's theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA$$

2. (20pts) Verify Stokes' Theorem for the part of the surface $z = 1 - x^2 - y^2$ where $z \geq 0$, where $\mathbf{F}(x, y, z) = \langle 2xy, x, y + z\rangle$ (see section 17.2). Use the upward pointing normal.

Stokes Theorem: $\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$

∂S is a unit circle in the xy-plane

(oriented clockwise to achieve boundary orientation when the surface has upward-pointing orientation)

$x = \cos t$
$y = \sin t$
$z = 0$

$$\int_{\partial S} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \langle 2\cos t \sin t, \cos t, \sin t + 0\rangle \cdot \langle -\sin t, \cos t, 0\rangle$$

$$= \int_0^{2\pi} -2\cos t \sin^2 t + \cos^2 t\,dt = \begin{bmatrix} u = \sin t & t = 2\pi,\ u = 0 \\ du = \cos t\,dt & t = 0,\ u = 0 \end{bmatrix}$$

on first term

$$= \underbrace{\int_0^0 -2u^2\,du}_{0} + \int_0^{2\pi} \tfrac{1}{2}(1 + \cos 2t)\,dt$$

gives 0 in $\int_0^{2\pi}$

$$= \frac{1}{2}\cdot 2\pi = \pi$$

Parametrize surface:

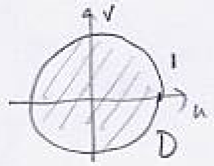
$x = u$

$y = v$

$z = 1 - u^2 - v^2$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = \langle 2u, 2v, 1 \rangle$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} = \langle 1-0, 0-0, 1-2x \rangle = \langle 1, 0, 1-2x \rangle$$



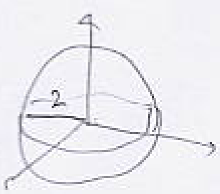
$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle 1, 0, 1-2u \rangle \cdot \langle 2u, 2v, 1 \rangle \, dA$$

$$= \iint_D 2u + 1 - 2u \, dA = \iint_D 1 \, dA = \text{area}(D) = \pi \cdot 1^2 = \pi$$

(got same answer)

3. (20pts) Verify the Divergence Theorem for the sphere $x^2 + y^2 + z^2 = 4$ and the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ (see section 17.3).

Divergence Theorem: $\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \text{div}\,\vec{F} \, dV$ (outward normal orientation)



$$T_\phi \times T_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

Parametrize sphere

$x = 2\sin\phi\cos\theta$

$y = 2\sin\phi\sin\theta$

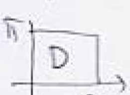
$z = 2\cos\phi$

$\theta \in [0, 2\pi]$

$\phi \in [0, \pi]$

$$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, \underbrace{4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta}_{4\sin\phi\cos\phi} \rangle$$

$$= 4\sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$



$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iint_D \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle \cdot 4\sin\phi \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle \, dA$$

$$= \iint_D 4\sin\phi \big(\underbrace{2\sin^2\phi\cos^2\theta + 2\sin^2\phi\sin^2\theta}_{2\sin^2\phi} + 2\cos^2\phi \big) dA = \iint_D 4\sin\phi \cdot 2 \, dA$$

$$= 8 \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta = 16\pi \cdot (-\cos\phi) \Big|_0^{\pi} = -16\pi(-1-1) = 32\pi$$

$\text{div}\,\vec{F} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y + \frac{\partial}{\partial z} z = 3$

$$\iiint_W 3 \, dV = 3\,\text{vol}(W) = 3 \cdot \frac{4\pi}{3} \cdot 2^3 = 32\pi \longrightarrow \text{equal!}$$