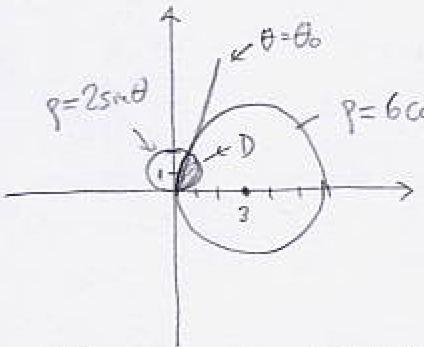
Fall '10/MAT 309/Worksheet 4 **Name:** Solution *Show all your work.*

1. (~~20~~ 25 pts) Find the centroid of the region that is common to circles $(x-3)^2+y^2=9$ and $x^2+(y-1)^2=1$. You can reduce your work by noticing that this region is symmetric about the line going through the center of the circles.



Need: $\iint_D x\,dA$, $\iint_D 1\,dA$

Line through the centers:

$\frac{x}{3}+\frac{y}{1}=1$

$y=-\frac{x}{3}+1$

Area of D = Area(D_1) + Area D_2

$$=\iint_{D_1} 1\,dA+\iint_{D_2} 1\,dA$$

$$=\int_0^{\theta_0}\int_0^{2\sin\theta} r\,dr\,d\theta+\int_{\theta_0}^{\frac{\pi}{2}}\int_0^{6\cos\theta} r\,dr\,d\theta$$

$$=\int_0^{\theta_0}\frac{1}{2}r^2\Big|_0^{2\sin\theta}d\theta+\int_{\theta_0}^{\pi/2}\frac{1}{2}r^2\Big|_0^{6\cos\theta}d\theta$$

$$=\int_0^{\theta_0}\frac{1}{2}4\sin^2\theta\,d\theta+\int_{\theta_0}^{\pi/2}\frac{1}{2}36\cos^2\theta\,d\theta$$

$$=\int_0^{\theta_0}2\cdot\frac{1-\cos 2\theta}{2}d\theta+\int_{\theta_0}^{\pi/2}18\cdot\frac{1+\cos 2\theta}{2}d\theta$$

$$=\theta_0-\frac{\sin 2\theta}{2}\Big|_0^{\theta_0}+9\left(\frac{\pi}{2}-\theta_0\right)+9\frac{\sin 2\theta}{2}\Big|_{\theta_0}^{\pi/2}$$

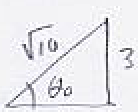
$$=\frac{9\pi}{2}-8\theta_0-\frac{1}{2}\sin 2\theta_0+\frac{9}{2}(0-\sin 2\theta_0)$$

$$=\frac{9\pi}{2}-8\theta_0-5\underbrace{\sin 2\theta_0}_{2\sin\theta_0\cos\theta_0\,=\,2\cdot\frac{1}{\sqrt{10}}\cdot\frac{3}{\sqrt{10}}}=\frac{9\pi}{2}-8\arctan 3-3=1.1448$$

Intersection: $2\sin\theta=6\cos\theta$

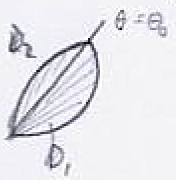
$\tan\theta=3$

$\theta_0=\arctan 3$



$\tan\theta_0=3$, $\cos\theta_0=\frac{1}{\sqrt{10}}$, $\sin\theta_0=\frac{3}{\sqrt{10}}$

D breaks up into two regions suitable for polar coordinates:



$$x_{CM}=\frac{27\left(\frac{\pi}{2}-\arctan 3-\frac{3}{10}\right)}{\frac{9\pi}{2}-8\arctan 3-3}$$

$$=0.5130$$

$$y_{CM}=-\frac{x_{CM}}{3}+1$$

$$=0.8290$$

$$=-\frac{9\left(\frac{\pi}{2}-\arctan 3-\frac{3}{10}\right)}{\frac{9\pi}{2}-8\arctan 3-3}+1$$

$$=\frac{\arctan 3-\frac{3}{10}}{\frac{9\pi}{2}-8\arctan 3-3}$$

2. (~~20~~ 15 pts) Verify our change-of-variables formula for spherical coordinates by computing the Jacobian $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}$. What should you get?

$x=\rho\sin\phi\cos\theta$

$y=\rho\sin\phi\sin\theta$

$z=\rho\cos\phi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix}\sin\phi\cos\theta & \rho\sin\phi(-\sin\theta) & \rho\cos\phi\cos\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta\\ \cos\phi & 0 & -\rho\sin\phi\end{vmatrix}=$$ expand by bottom row

$$=\cos\phi\left(-\rho^2\sin\phi\cos\phi\sin^2\theta-\rho^2\sin\phi\cos\phi\cos^2\theta\right)-\rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta+\rho\sin^2\phi\sin^2\theta\right)$$

$$=-\rho^2\sin\phi\cos^2\phi\underbrace{(\sin^2\theta+\cos^2\theta)}_{=1}-\rho^2\sin^3\phi\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}$$

$$=-\rho^2\sin\phi(\cos^2\phi+\sin^2\phi)=-\rho^2\sin\phi$$

$|-\rho^2\sin\phi|=\rho^2\sin\phi$, the term used in changing integration to spherical coordinates

$$\iint_{D_1} x\,dA + \iint_{D_2} x\,dA$$

$$= \int_0^{\theta_0}\int_0^{2\sin\theta} r\cos\theta\, r\,dr\,d\theta + \int_{\theta_0}^{\pi/2}\int_0^{6\cos\theta} r\cos\theta\, r\,dr\,d\theta$$

$$= \int_0^{\theta_0} \cos\theta\, \frac13 r^3\Big|_0^{2\sin\theta} + \int_{\theta_0}^{\pi/2}\cos\theta\,\frac13 r^3\Big|_0^{6\cos\theta} d\theta$$

$$= \frac83\int_0^{\theta_0}\cos\theta\sin^3\theta\, d\theta + 72\int_{\theta_0}^{\pi/2}\cos^4\theta\, d\theta$$

$$= \begin{bmatrix} u=\sin\theta & \theta=\theta_0,\ u=\frac{3}{\sqrt{10}} \\ du=\cos\theta\, d\theta & \theta=0,\ u=0 \end{bmatrix}$$

$$= \frac83\int_0^{\frac{3}{\sqrt{10}}} u^3\, du + 72\int_{\theta_0}^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)^2 d\theta$$

$$= \frac83\, \frac{u^4}{4}\Big|_0^{\frac{3}{\sqrt{10}}} + \frac{72}{4}\int_{\theta_0}^{\pi/2} 1+2\cos2\theta+\cos^2 2\theta\, d\theta$$

$$= \frac23\cdot\frac{81}{100} + 18\int_{\theta_0}^{\pi/2} 1+2\cos2\theta+\frac{1+\cos4\theta}{2}\,d\theta$$

$$= \frac{27}{50}+18\int_{\theta_0}^{\pi/2}\frac32+2\cos2\theta+\frac12\cos4\theta\, d\theta$$

$$= \frac{27}{50}+18\left(\frac32\left(\frac\pi2-\theta_0\right)+2\cdot\frac{\sin2\theta}{2}\Big|_{\theta_0}^{\pi/2}+\frac12\,\frac{\sin4\theta}{4}\Big|_{\theta_0}^{\pi/2}\right)$$

$$= \frac{27}{50}+27\left(\frac\pi2-\theta_0\right)-18\sin2\theta_0-\frac94\sin4\theta_0$$

$$= \frac{27}{50}+27\left(\frac\pi2-\theta_0\right)-18\cdot2\sin\theta_0\cos\theta_0-\frac94\cdot2\sin2\theta_0\cos2\theta_0$$

$$= \frac{27}{50}+27\left(\frac\pi2-\theta_0\right)-36\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}-\frac92\cdot2\sin\theta_0\cos\theta_0\left(\cos^2\theta_0-\sin^2\theta_0\right)$$

$$= \frac{27}{50}+27\left(\frac\pi2-\theta_0\right)-\frac{108}{10}-9\cdot\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{10}}\left(\frac{1}{10}-\frac{9}{10}\right)$$

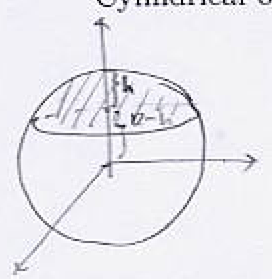
$$= 27\left(\frac\pi2-\theta_0\right)+\frac{27}{50}-\frac{540}{50}+\frac{27\cdot4}{50} = 27\left(\frac\pi2-\theta_0\right)-\frac{27\cdot15}{50}$$

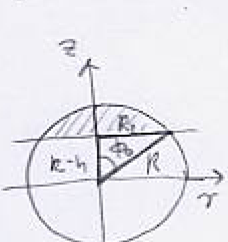
$$= 27\left(\frac\pi2-\arctan 3-\frac{3}{10}\right)\simeq 0.5873$$

$1-20+4$

3. (20pts) A spherical cap of height h is the set $x^2+y^2+z^2\le R^2$, $z\ge R-h$. Show that its volume is $V=\frac13\pi h^2(3R-h)$. Then use this formula to get the volume of a ball or radius R. Cylindrical or spherical coordinates may be useful here.

half-ball is when $h=R$. Volume of ball is $V=2\cdot\frac13\pi R^2(3R-R)=\frac{4\pi}{3}R^2$





$\cos\phi_0=\frac{R-h}{R}$

$R_1^2+(R-h)^2=R^2$

$z=R-h$

$\rho\cos\phi=R-h$

$\rho=\frac{R-h}{\cos\phi}$

Cylindrical coordinates:

$$V=\iiint_W 1\,dV = \int_0^{2\pi}\int_0^{R_1}\int_{R-h}^{\sqrt{R^2-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

$$=2\pi\int_0^{R_1} r\left(\sqrt{R^2-r^2}-R+h\right)dr$$

$$=2\pi\left[-\frac12\,\frac23\left(R^2-r^2\right)^{\frac32}\Big|_0^{R_1}+(h-R)\frac{r^2}{2}\Big|_0^{R_1}\right]$$

$$=2\pi\left[-\frac13\left(\left(R^2-R_1^2\right)^{\frac32}-R^3\right)+(h-R)\cdot\frac{R_1^2}{2}\right]$$

$$=\frac13\pi\left(2\left(R^3-(R-h)^3\right)+3(h-R)\left(R^2-(R-h)^2\right)\right)$$

$$=\frac13\pi\left(2R^3-2(R-h)^3+3(h-R)R^2+3(R-h)^3\right)$$

$$=\frac13\pi\left((R-h)^3+3hR^2-3R^3+2R^3\right)$$

$$=\frac13\pi\left(R^3-3R^2h+3Rh^2-h^3+3R^2h-R^3\right)$$

$$=\frac13\pi h^2(3R-h)$$

Spherical coordinates:

$$V=\int_0^{2\pi}\int_0^{\phi_0}\int_{\frac{R-h}{\cos\phi}}^{R} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=2\pi\int_0^{\phi_0}\sin\phi\,\frac{\rho^3}{3}\Big|_{\frac{R-h}{\cos\phi}}^{R} d\phi$$

$$=\frac{2\pi}{3}\int_0^{\phi_0}\sin\phi\left(R^3-\frac{(R-h)^3}{\cos^3\phi}\right)d\phi = \begin{bmatrix} u=\cos\phi \\ du=-\sin\phi\, d\phi \\ \phi=\phi_0,\ u=\frac{R-h}{R} \\ \phi=0,\ u=1 \end{bmatrix}$$

$$=\frac{2\pi}{3}\int_1^{\frac{R-h}{R}}\left(R^3-\frac{(R-h)^3}{u^3}\right)(-du)=\frac{2\pi}{3}\int_{\frac{R-h}{R}}^1 R^3-\frac{(R-h)^3}{u^3}\,du$$

$$=\frac{2\pi}{3}\left(R^3\left(1-\frac{R-h}{R}\right)-(R-h)^3\left(-\frac12\right)\frac{1}{u^2}\Big|_{\frac{R-h}{R}}^1\right)$$

$$=\frac{\pi}{3}\left(2R^3\cdot\frac{h}{R}+(R-h)^3\left(1-\frac{R^2}{(R-h)^2}\right)\right)$$

$$=\frac{\pi}{3}\left(2R^2h+(R-h)^3-R^2(R-h)\right)$$

$$=\frac{\pi}{3}\left(2R^2h+R^3-3R^2h+3Rh^2-h^3-R^3+R^2h\right)$$

$$=\frac{\pi}{3}h^2(3R-h)$$