Fall '10/MAT 309/Worksheet 3 **Name:** Solution *Show all your work.*

1. (20pts) An unfortunate twist of events finds you near the bottom of a snake pit described by the surface $z = \frac{x^2}{4} + \frac{y^2}{9} - 10$. You are at point $(2, 3, -8)$ and there is a caucus of rattlesnakes around the point $(0, 0, -10)$. You wish to move away from the snakes by taking the path of steepest ascent. Note that this is really a two-dimensional problem. Let $f(x, y) = \frac{x^2}{4} + \frac{y^2}{9} - 10$.
a) Draw the level curves of f for levels $c = -9, -8, -6, -1$.
b) From point $(2, 3)$, in which direction should you start going to achieve the greatest increase of f? Using the level curves you drew in a), draw an approximate path that always goes in the direction of greatest increase (this is the projection of your escape path to the xy plane).
c) Show that the path given by $\mathbf{c}(t) = \langle 2t^9, 3t^4 \rangle$ is such a path, that is, show that for every t, $\mathbf{c}'(t)$ is parallel to $\nabla f(\mathbf{c}(t))$.

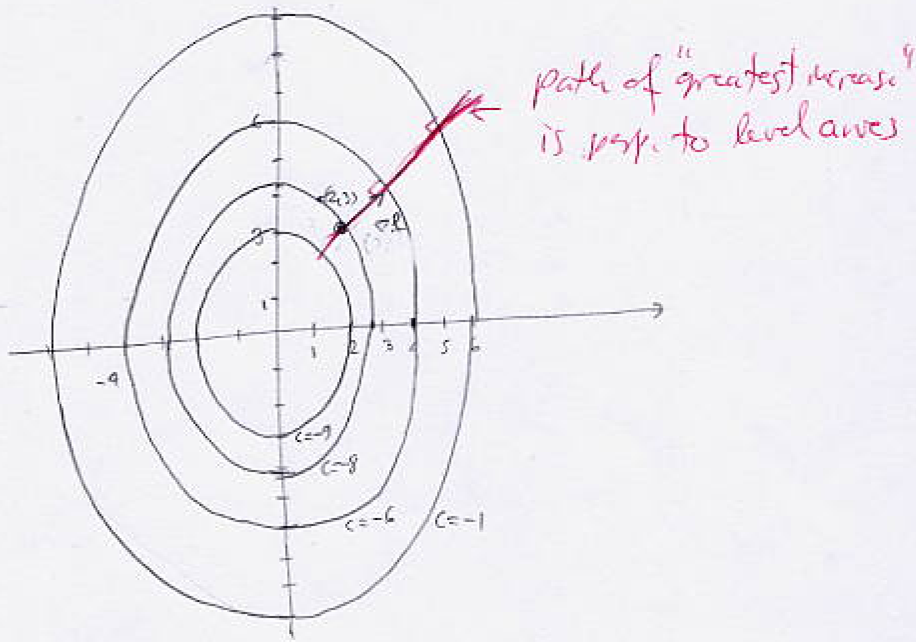
a) $\frac{x^2}{4} + \frac{y^2}{9} - 10 = \begin{cases} -9 \\ -8 \\ -6 \\ -1 \end{cases}$ all ellipses

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ellipse, $a=2,\ b=3$

$\left(\frac{x^2}{8} + \frac{y^2}{18} = 1\right)$ $\frac{x^2}{4} + \frac{y^2}{9} = 2$ $a = 2\sqrt{2}$ $b = 3\sqrt{2}$

$\frac{x^2}{16} + \frac{y^2}{36} = 1$ $\frac{x^2}{4} + \frac{y^2}{9} = 4$ $a=4,\ b=6$

$\frac{x^2}{36} + \frac{y^2}{81} = 1$ $\frac{x^2}{4} + \frac{y^2}{9} = 9$ $a=6,\ b=9$



b) In the direction of $\nabla f(2,3)$

$\nabla f = \left\langle \frac{2x}{4}, \frac{2y}{9} \right\rangle = \left\langle \frac{x}{2}, \frac{2y}{9} \right\rangle$

$\nabla f(2,3) = \left\langle 1, \frac{2}{3} \right\rangle$

c) $\vec{c}(t) = \langle 2t^9, 3t^4 \rangle$

(Note: $\vec{c}(1) = \langle 2, 3 \rangle$)

$\vec{c}\,'(t) = \langle 18t^8, 12t^3 \rangle =$

$\nabla f(2t^9, 3t^4) = \left\langle \frac{2t^9}{2}, \frac{2\cdot 3t^4}{9} \right\rangle = \left\langle t^9, \frac{2}{3}t^4 \right\rangle = t\left\langle t^8, \frac{2}{3}t^3 \right\rangle = \frac{t}{18}\langle 18t^8, 12t^3 \rangle = \frac{t}{18}\vec{c}\,'(t)$

Thus, $\nabla f(\vec{c}(t))$ is parallel to $\vec{c}\,'(t)$.

(2) $x = r\cos\theta$
$y = r\sin\theta$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{\partial}{\partial r}\frac{\partial f}{\partial r} = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial y}\right)\sin\theta = \left[\quad\right]$$

$$= \left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial y}{\partial r}\right)\cdot\cos\theta + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right)\sin\theta$$

$$= \left(\frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial y\partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 f}{\partial x\partial y}\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin\theta\right)\sin\theta$$

$$= \boxed{\frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial y\partial x}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta}$$

$$\frac{\partial^2 f}{\partial\theta\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) = \frac{\partial}{\partial\theta}\frac{\partial f}{\partial x}\cos\theta + \boxed{\frac{\partial f}{\partial x}(-\sin\theta)} + \frac{\partial}{\partial\theta}\frac{\partial f}{\partial y}\sin\theta + \boxed{\frac{\partial f}{\partial y}\cos\theta}$$

$$= \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\frac{\partial y}{\partial\theta}\right)\cos\theta + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta}\right)\sin\theta + \boxed{\frac{\partial f}{\partial y}\cos\theta - \frac{\partial f}{\partial x}\sin\theta}$$

$$= \left(\frac{\partial^2 f}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 f}{\partial y\partial x}r\cos\theta\right)\cos\theta + \left(\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 f}{\partial y^2}r\cos\theta\right)\sin\theta + \frac{\partial f}{\partial y}\cos\theta - \frac{\partial f}{\partial x}\sin\theta$$

$$\boxed{= \frac{\partial^2 f}{\partial y\partial x}\cdot r(\cos^2\theta - \sin^2\theta) + r\sin\theta\cos\theta\left(\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2}\right) + \frac{\partial f}{\partial y}\cos\theta - \frac{\partial f}{\partial x}\sin\theta}$$

$$\frac{\partial^2 f}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(-\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta\right) = -\left(\frac{\partial}{\partial\theta}\frac{\partial f}{\partial x}r\sin\theta + \boxed{\frac{\partial f}{\partial x}r\cos\theta}\right) + \frac{\partial}{\partial\theta}\frac{\partial f}{\partial y}r\cos\theta + \boxed{\frac{\partial f}{\partial y}(-r\sin\theta)}$$

$$= -\left(\frac{\partial}{\partial x}\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\frac{\partial y}{\partial\theta}\right)r\sin\theta + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta}\right)r\cos\theta \boxed{-\frac{\partial f}{\partial x}r\cos\theta - \frac{\partial f}{\partial y}r\sin\theta}$$

$$= -\left(\frac{\partial^2 f}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 f}{\partial y\partial x}r\cos\theta\right)r\sin\theta + \left(\frac{\partial^2 f}{\partial x\partial y}(-r\sin\theta) + \frac{\partial^2 f}{\partial y^2}r\cos\theta\right)r\cos\theta - \frac{\partial f}{\partial x}r\cos\theta - \frac{\partial f}{\partial y}r\sin\theta$$

$$\boxed{= \frac{\partial^2 f}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2 f}{\partial x\partial y}r^2\sin\theta\cos\theta + \frac{\partial^2 f}{\partial y^2}r^2\cos^2\theta - \frac{\partial f}{\partial x}r\cos\theta - \frac{\partial f}{\partial y}r\sin\theta}$$

Ugh!

(See exercise 41 in 14.6 for a more general statement)

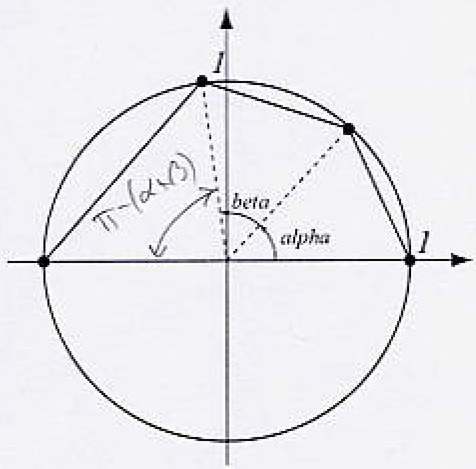
2. (20pts) Let $x = r\cos\theta$, $y = r\sin\theta$ and let $f(x, y)$ be a function. Express $\frac{\partial^2 f}{\partial r^2}$, $\frac{\partial^2 f}{\partial\theta\partial r}$ and $\frac{\partial^2 f}{\partial\theta^2}$ in terms of r, θ and first and second partial derivatives of f by x and y. It's best if you do this on a separate sheet of paper, mwahahahahaha....

3. (20pts) Among all quadrangles inscribed in the unit circle whose one side is a diameter, find the one with the greatest area. Do it as follows:

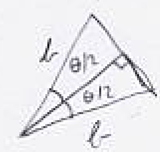
a) Express the area A of the quadrangle as a function of angles α and β.

b) Determine the restrictions on α and β — this gives you the domain.

c) Find the maximum of A over the domain in the usual way.



a) Area of an isosceles triangle



$A = \frac{1}{2} l\sin\frac{\theta}{2}\, l\cos\frac{\theta}{2}$

$= \frac{1}{2} l^2 \sin\left(2\cdot\frac{\theta}{2}\right) = \frac{1}{2} l^2 \sin\theta$

In our case, $l = 1$.

$A(\alpha,\beta) = \frac{1}{2}\sin\alpha + \frac{1}{2}\sin\beta + \frac{1}{2}\sin(\pi - (\alpha+\beta))$

$= \frac{1}{2}\left(\sin\alpha + \sin\beta + \sin(\alpha+\beta)\right)$

(Recall $\sin(\pi-\theta) = \sin\theta$)

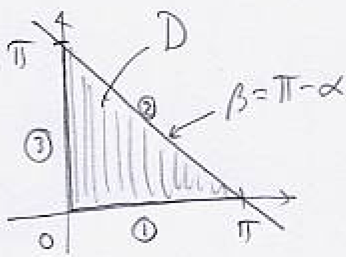
b) Must have:

$0 \le \alpha \le \pi$

$0 \le \beta \le \pi$

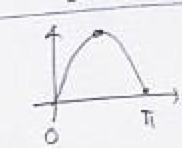
$\alpha+\beta \le \pi$

$(\beta \le \pi - \alpha)$



Job: maximize $A(\alpha,\beta)$ over region with boundary D.

Boundary: ① $\alpha = t$, $\beta = 0$, $t\in[0,\pi]$; $A(t,0) = \sin t$



max for $t = \frac{\pi}{2}$

② $\alpha = t$, $\beta = \pi - t$, $t\in[0,\pi]$; $A(t,\pi-t) = \frac{1}{2}(\sin t + \sin(\pi - t)) = \sin t$, max for $t = \frac{\pi}{2}$

③ $\alpha = 0$, $\beta = t$, $t\in[0,\pi]$; $A(0,t) = \sin t$, max for $t = \frac{\pi}{2}$

c) Critical points:

$A_\alpha = \frac{1}{2}(\cos\alpha + \cos(\alpha+\beta))$

$A_\beta = \frac{1}{2}(\cos\beta + \cos(\alpha+\beta))$

$\begin{cases}\cos\alpha + \cos(\alpha+\beta) = 0\\ \cos\beta + \cos(\alpha+\beta) = 0\end{cases}$

$\Rightarrow \cos\alpha = -\cos(\alpha+\beta) = \cos\beta$

Since $\alpha,\beta\in[0,\pi]$, where cos is one-to-one, this means $\alpha = \beta$

$\cos\alpha + \cos(2\alpha) = 0$

$\cos\alpha + 2\cos^2\alpha - 1 = 0$

$2\cos^2\alpha + \cos\alpha - 1 = 0$

$\cos\alpha = \dfrac{-1 \pm\sqrt{1^2 - 4\cdot2\cdot(-1)}}{2\cdot 2}$

$= \dfrac{-1\pm\sqrt{9}}{4} = -1, \frac{1}{2}$

$\alpha = \pi$ or $\alpha = \frac{\pi}{3}$

so $\beta = \pi$ $\quad\beta = \frac{\pi}{3}$

(π,π) not in region D $\quad\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ in region

(α,β)	$A(\alpha,\beta)$
$\left(\frac{\pi}{3},\frac{\pi}{3}\right)$	$\frac{1}{2}\left(\frac{\sqrt3}{2}+\frac{\sqrt3}{2}+\frac{\sqrt3}{2}\right) = \frac{3\sqrt3}{4} \approx 1.30$
$\left(\frac{\pi}{2},0\right)$	$\frac{1}{2}(1+0+1) = 1$
$\left(\frac{\pi}{2},\frac{\pi}{2}\right)$	$\frac{1}{2}(1+1+0) = 1$
$\left(0,\frac{\pi}{2}\right)$	$\frac{1}{2}(0+1+1) = 1$

max occurs at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

red with biggest area