Fall '10/MAT 309/Worksheet 1 **Name:** Solution *Show all your work.*

1. (12pts) Find all vectors whose angles with **i** and **k** are $\frac{\pi}{3}$ and $\frac{\pi}{4}$, respectively. *(Hint: look for unit vectors with unknown coordinates $\langle a, b, c\rangle$).*

We are looking for a $\vec{v} = \langle a, b, c\rangle$

so that: $\|\vec{v}\| = 1$ $\qquad \sqrt{a^2+b^2+c^2} = 1$

$\cos\frac{\pi}{3} = \frac{\vec{v}\cdot\vec{i}}{\|\vec{v}\|\,\|\vec{i}\|} \qquad \frac{1}{2} = a$

$\cos\frac{\pi}{4} = \frac{\vec{v}\cdot\vec{k}}{\|\vec{v}\|\,\|\vec{k}\|} \qquad \frac{\sqrt{2}}{2} = c$

So: $\left(\frac{1}{2}\right)^2 + b^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$

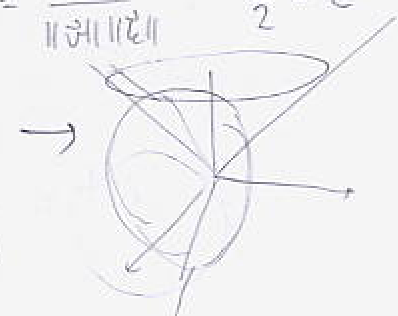
$\frac{1}{4} + b^2 + \frac{1}{2} = 1$

$b^2 + \frac{3}{4} = 1$

$b^2 = \frac{1}{4}$

$b = \pm\frac{1}{2}$

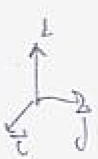
solution are the intersection of two cones $\rightarrow$



The vectors are $\left\langle \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right\rangle$

and $\left\langle \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right\rangle$

2. (18pts) a) Find three vectors **u**, **v** and **w** for which $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) \neq (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$. (Use really simple vectors.)
b) Let **u**, **v** and **w** be any vectors. Note that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ is a vector that is in the plane spanned by **v** and **w**, since it is perpendicular to $\mathbf{v}\times\mathbf{w}$, which is a normal vector for the plane spanned by **v** and **w**. Therefore, $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ can be written as $\beta\mathbf{v}+\gamma\mathbf{w}$. Use coordinates to show that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$.



a) $(\vec{i}\times\vec{i})\times\vec{j} = \vec{0}\times\vec{j} = \vec{0}$
$\vec{i}\times(\vec{i}\times\vec{j}) = \vec{i}\times\vec{k} = -\vec{j}$ } not equal

$\vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \langle b_2c_3 - b_3c_2, -(a_2c_3 - a_3c_2), a_2b_3 - a_3b_2\rangle$

b) Let $\vec{u} = \langle a_1, b_1, c_1\rangle$
$\vec{v} = \langle a_2, b_2, c_2\rangle$
$\vec{w} = \langle a_3, b_3, c_3\rangle$

$\vec{u}\times(\vec{v}\times\vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_1 & c_1 \\ b_2c_3 - b_3c_2 & a_3c_2 - a_2c_3 & a_2b_3 - a_3b_2 \end{vmatrix}$

$\vec{v}\times w$

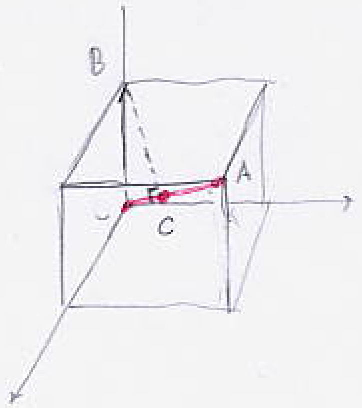
$= \langle b_1(a_2b_3 - a_3c_2) - c_1(a_3c_2 - a_2c_3), -(a_1(a_2b_3 - a_3b_2) - c_1(b_2c_3 - b_3c_2)),$
$a_1(a_3c_2 - a_2c_3) - b_1(b_2c_3 - b_3c_2)\rangle$

$= \langle a_2(b_1b_3 + c_1c_3) - a_3(b_1b_2 + c_1c_2), b_2(a_1a_3 + c_1c_3) - b_3(a_1a_2 + c_1c_2),$
$c_2(a_1a_3 + b_1b_3) - c_3(a_1a_2 + b_1b_2)\rangle$

$= \langle a_2\overbrace{(a_1a_3 + b_1b_3 + c_1c_3)}^{\vec{u}\cdot\vec{w}} - a_3\overbrace{(a_1a_2 + b_1b_2 + c_1c_2)}^{\vec{u}\cdot\vec{v}}, b_2\overbrace{(a_1a_3 + b_1b_3 + c_1c_3)}^{\vec{u}\cdot\vec{w}} - b_3\underbrace{(a_1a_2 + b_1b_2 + c_1c_2)}_{\vec{u}\cdot\vec{v}}, c_2\underbrace{(a_1a_3 + b_1b_3 + c_1c_3)}_{\vec{u}\cdot\vec{w}} - c_3\underbrace{(a_1a_2 + b_1b_2 + c_1c_2)}_{\vec{u}\cdot\vec{v}}$

$= (\vec{u}\cdot\vec{w})\langle a_2, b_2, c_2\rangle - (\vec{u}\cdot\vec{v})\langle a_3, b_3, c_3\rangle = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$

3. (12pts) Let a cube be positioned so that one of its vertices is at the origin and three of its edges are along the positive x, y and z- axes. Let $A = (1, 1, 1)$ be a vertex of the cube. Use projection of vectors to find the distance from the vertex $B = (0, 0, 1)$ to the diagonal OA.



Need the projection of $\vec{OB}=\langle 0,0,1\rangle$ to $\vec{OA}=\langle 1,1,1\rangle$

$$\text{proj}_{\vec{OA}}\vec{OB} = \frac{\vec{OB}\cdot\vec{OA}}{\|\vec{OA}\|^2}\vec{OA} = \frac{1}{3}\langle 1,1,1\rangle = \langle \tfrac13,\tfrac13,\tfrac13\rangle$$

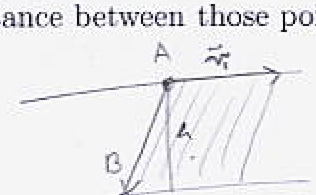
Thus, $C=(\frac13,\frac13,\frac13)$

Distance from B to diagonal $= d(B,C)$

$$= \sqrt{(0-\tfrac13)^2+(0-\tfrac13)^2+(1-\tfrac13)^2} = \sqrt{\tfrac19+\tfrac19+\tfrac49} = \sqrt{\tfrac69} = \sqrt{\tfrac23}$$

4. (18pts) Two parallel lines are given parametrically: $x = 1-t$, $y = 4+2t$, $z = 3+2t$ and $x = 2t$, $y = 1-4t$, $z = -3-4t$.
Find the distance between those lines in two ways (you'd better get the same answer!):
a) Use the height and area of a parallelogram (draw a picture).
b) Find a plane perpendicular to the two lines that passes through a known point on one line. The intersection of the plane with the other line will give you another point. Find the distance between those points.

a)



Area $= \|\vec{v}_1\|\cdot h$

$\|\vec{v}_1\times\vec{AB}\| = \|\vec{v}_1\|\, h$

$\vec{v}_1=\langle -1,2,2\rangle$ $\vec{AB}=\langle -1,-3,-6\rangle$

$$\vec{v}_1\times\vec{AB} = \begin{vmatrix} \vec i & \vec j & \vec k \\ -1 & 2 & 2 \\ -1 & -3 & -6 \end{vmatrix} = -6\vec i - 8\vec j + 5\vec k$$

$\|\vec{v}_1\times\vec{AB}\| = \sqrt{36+64+25} = \sqrt{125} = 5\sqrt5$

$\|\vec{v}_1\| = \sqrt{1+4+4} = 3$ $\quad 5\sqrt5 = 3h$

$h = \frac{5\sqrt5}{3}$ $\quad \xleftrightarrow{\text{got same answer}}$

b) Plane through $(1,4,3)$, perpendicular to first line:
(normal vector = direction vector $\vec{v}_1$)

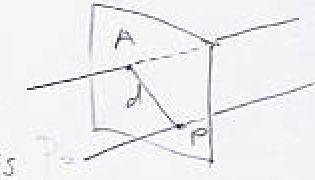
$-x+2y+2z = -1+2\cdot4+2\cdot3$

$-x+2y+2z = 13$ intersect with second line:

$-2t+2(1-4t)+2(-3-4t)=13$

$-2t-8t-8t+4=13$

$-18t=17$ $\quad t=-\frac{17}{18}$, point is



$P=\left(2\cdot(-\frac{17}{18}),\ 1-4\cdot(-\frac{17}{18}),\ -3-4\cdot(-\frac{17}{18})\right) = \left(-\frac{17}{9},\frac{43}{9},\frac{7}{9}\right)$

$$d(A,P) = \sqrt{\left(-\tfrac{17}{9}-1\right)^2+\left(\tfrac{43}{9}-4\right)^2+\left(\tfrac{7}{9}-3\right)^2}$$

$$= \sqrt{\left(\tfrac{26}{9}\right)^2+\left(\tfrac{7}{9}\right)^2+\left(\tfrac{20}{9}\right)^2} = \sqrt{\frac{676+49+400}{81}} = \frac{\sqrt{\overset{9\cdot125}{1125}}}{9} = \frac{3\sqrt{125}}{9} = \frac{5\sqrt5}{3}$$