Fall '10/MAT 309/Final Exam **Name:** Solution *Show all your work.*

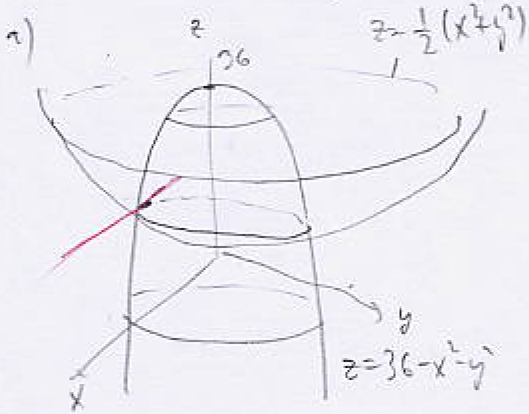
1. (6pts) Let $\mathbf{u} = \langle 3, 1, -7\rangle$ and $\mathbf{v} = \langle 2, -3, 1\rangle$. Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{6-3-7}{\sqrt{9+1+49}\sqrt{4+9+1}} = -\frac{4}{\sqrt{59}\sqrt{14}}$$

$$\theta = \arccos\left(-\frac{4}{\sqrt{59}\cdot\sqrt{14}}\right)$$

2. (16pts) The paraboloids $z = \frac{1}{2}(x^2+y^2)$ and $z = 36 - x^2 - y^2$ intersect in a circle.
a) Sketch a picture.
b) Find a parametrization for the circle.
c) Find the parametric equation of the tangent line to the circle at point $(3\sqrt{2}, -\sqrt{6}, 12)$ and draw it on your sketch.

a)



$\frac{1}{2}(x^2+y^2) = 36 - x^2 - y^2$

$\frac{3}{2}(x^2+y^2) = 36 \quad \big| \cdot \frac{2}{3}$

$x^2+y^2 = 24$

Circle is on cylinder of radius $\sqrt{24} = 2\sqrt{6}$

$z = \frac{1}{2}\cdot 24 = 12$

Parametrization is

$x = 2\sqrt{6}\cos t$
$y = 2\sqrt{6}\sin t$
$z = 12$

c) $2\sqrt{6}\cos t = 3\sqrt{2}$

$\cos t = \frac{3}{2}\cdot\sqrt{\frac{2}{6}} = \frac{3}{2}\sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{2}$

$t = \frac{\pi}{6}, -\frac{\pi}{6}$

$t = -\frac{\pi}{6}$ since $2\sqrt{6}\sin t = -\sqrt{6}$ (negative

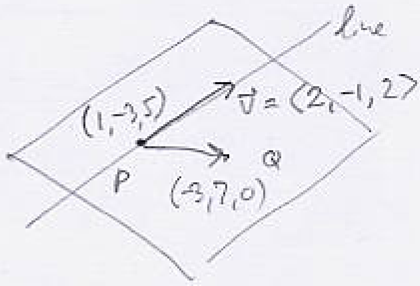
$\vec{c}\,'(t) = \langle -2\sqrt{6}\sin t, 2\sqrt{6}\cos t, 0\rangle$

$\vec{c}\,'(-\frac{\pi}{6}) = \langle -2\sqrt{6}\cdot\frac{1}{2}, 2\sqrt{6}\cdot\frac{\sqrt{3}}{2}, 0\rangle$

$= \langle -\sqrt{6}, 3\sqrt{2}, 0\rangle$

Tangent line: $x = 3\sqrt{2} - \sqrt{6}t$
$y = -\sqrt{6} + 3\sqrt{2}t$
$z = 12$

3. (10pts) A line is given parametrically: $x = 1 + 2t$, $y = -3 - t$, $z = 5 + 2t$. Find the equation of the plane that contains this line and the point $(-3, 7, 0)$.



Need $\vec{PQ} = \langle -4, 10, -5\rangle$

$\vec{n} = \vec{v} \times \vec{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ -4 & 10 & -5 \end{vmatrix} = \langle -15, -2, 16\rangle$ Use $\langle 15, 2, -16\rangle$

Equation of plane:

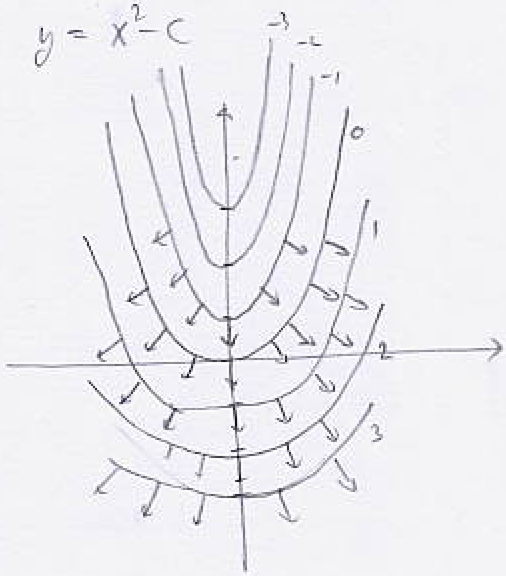
$15x - 2y - 16z = 15\cdot 1 - 2\cdot(-3) - 16\cdot 5$

$15x - 2y - 16z = -59$

4. (12pts) Let $f(x, y) = x^2 - y$.
a) Draw a rough contour map for the function with contour interval 1, going from $c = -3$ to $c = 3$.
b) Find ∇f and roughly draw this vector field. Note that no computation is needed to draw the vector field.
c) What is $\int_C \nabla f \cdot ds$ if C is the arc of the unit circle that is in the first quadrant, going counterclockwise?

a) $x^2 - y = c$

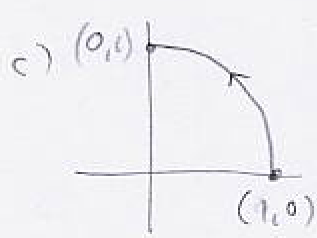
$y = x^2 - c$



b) $\nabla f = \langle 2x, -1\rangle$

∇f is perpendicular to level curves, points in direction of greater f

c)



$\int_C \nabla f \cdot d\vec{s} = f(0,1) - f(1,0)$

$= (0-1) - (1-0) = -2$

5. (16pts) Let $f(x,y) = ye^{xy}$, $x = u^3 - v^3$, $y = \frac{u}{v}$. Find $\dfrac{\partial f}{\partial v}$ when $u = 2$, $v = 1$.

$$\frac{\partial f}{\partial x} = ye^{xy}\cdot y = y^2 e^{xy}$$

$$\frac{\partial f}{\partial y} = 1\cdot e^{xy} + ye^{xy}\cdot x = (1+xy)\, e^{xy}$$

$$\frac{\partial x}{\partial v} = -3v^2$$

$$\frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

When $u=2$, $v=1$ we have $x=7$, $y=2$

$$\left.\frac{\partial f}{\partial v}\right|_{(2,1)} = 4e^{2\cdot 7}\cdot(-3\cdot 1^2) + (1+2\cdot 7)e^{2\cdot 7}\cdot\left(-\frac{2}{1^2}\right)$$

$$= -12e^{14} - 30e^{14} = -42e^{14}$$

6. (16pts) Find and classify the local extremes for the function $f(x,y) = x^3 - xy + y^3$.

$$\frac{\partial f}{\partial x} = 3x^2 - y$$

$$\frac{\partial f}{\partial y} = 3y^2 - x$$

$$\begin{cases} 3x^2 - y = 0 \Rightarrow y = 3x^2 \\ 3y^2 - x = 0 \end{cases}$$ put here

$$3(3x^2)^2 - x = 0$$

$$27x^4 - x = 0$$

$$x(27x^3 - 1) = 0$$

$x = 0$ or $x^3 = \frac{1}{27}$

$y = 0$ $\quad x = \frac{1}{3}$

$$y = 3\cdot\left(\frac{1}{3}\right)^2 = \frac{1}{3}$$

Candidates: $(0,0)$, $\left(\frac{1}{3}, \frac{1}{3}\right)$

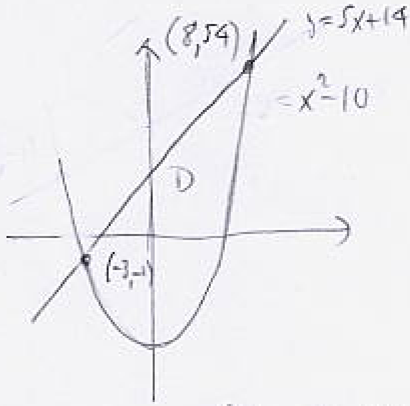
$$D = \begin{vmatrix} 6x & -1 \\ -1 & 6y \end{vmatrix}$$

$$D(0,0) = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} = -1 < 0$$ so it is a saddle point

$$D\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2 > 0$$ so f has local min at $\left(\frac{1}{3}, \frac{1}{3}\right)$

7. (18pts) Find $\iint_D x\,dA$ if D is the region bounded by the curves $y = x^2 - 10$ and $y = 5x + 14$.



Intersection: $x^2 - 10 = 5x + 14$

$x^2 - 5x - 24 = 0$

$(x-8)(x+3) = 0$

$x = 8, -3$

$$\iint_D x\,dA = \int_{-3}^{8}\int_{x^2-10}^{5x+14} x\,dy\,dx$$

$$= \int_{-3}^{8} x(5x+14-x^2+10)\,dx$$

$$= \int_{-3}^{8} x(5x - x^2 + 24)\,dx = \int_{-3}^{8} -x^3 + 5x^2 + 24x\,dx$$

$$= -\frac{x^4}{4}\Big|_{-3}^{8} + \frac{5}{3}x^3\Big|_{-3}^{8} + 24\frac{x^2}{2}\Big|_{-3}^{8}$$

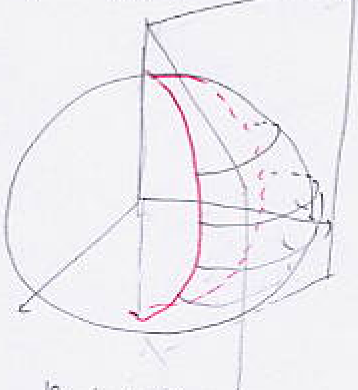
$$= -\frac{1}{4}(4096-81) + \frac{5}{3}(512-(-27)) + 12(64-9)$$

$$= \frac{1}{4}\cdot 4015 + \frac{5}{3}\cdot 539 + 660 = \frac{4015}{4} + \frac{2695}{3} + 660$$

$64 \cdot 8$ / $512 \cdot 8$ / 4096

$55 \cdot 12$ / 110 / 660

8. (16pts) Use cylindrical or spherical coordinates to set up $\iiint_W \dfrac{x^2+y^2+z^2}{x^2+y^2+1}\,dV$ where W is the region inside the sphere $x^2+y^2+z^2 \le 16$, between the planes $y = \sqrt{3}x$ and $y = -\frac{1}{\sqrt{3}}x$, and where $y \ge 0$. Sketch the region of integration. Do not evaluate the integral.



Region is a wedge in a ball

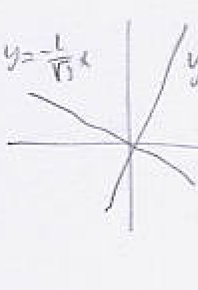
Radius = 4

Cylindrical

$$\int_{\pi/3}^{5\pi/6}\int_{0}^{4}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} \frac{r^2+z^2}{r^2+1}\, r\,dz\,dr\,d\theta$$

Spherical:

$$\int_{\pi/3}^{5\pi/6}\int_{0}^{\pi}\int_{0}^{4} \underbrace{\frac{\rho^2}{\rho^2\sin^2\phi+1}\,\rho^2\sin\phi}_{\frac{\rho^4\sin\phi}{\rho^2\sin^2\phi+1}}\,d\rho\,d\phi\,d\theta$$



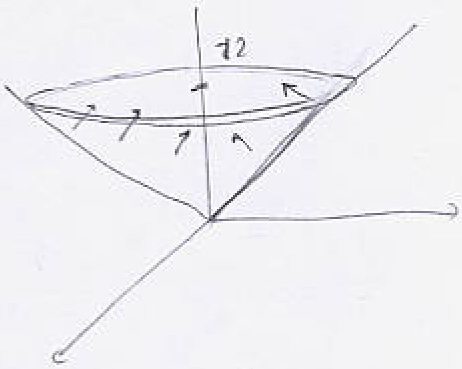
$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\theta = \frac{5\pi}{6}$

9. (24pts) Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$, if S is the part of the cone $z = 2\sqrt{x^2 + y^2}$ for which $z \leq 12$, and $\mathbf{F}(x, y, z) = \langle yz, xz, xy \rangle$. Use the normal vectors to the surface that point upwards. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from.



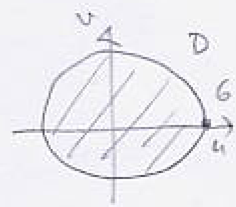
$2\sqrt{x^2+y^2} = 12$

$\sqrt{x^2+y^2} = 6$

Parametrization: $x = u$

$y = v$

$z = 2\sqrt{u^2+v^2}$



(projection to xy-plane)

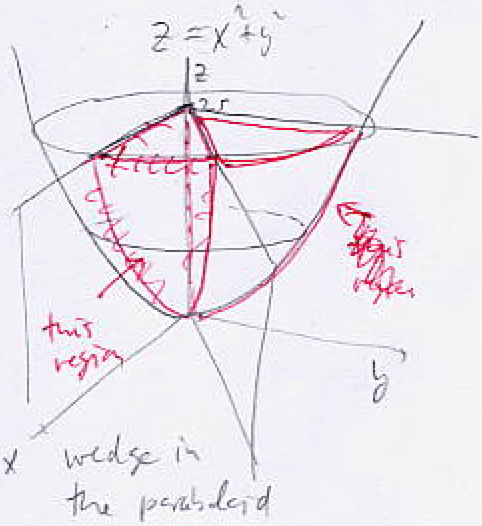
$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2\frac{2u}{2\sqrt{u^2+v^2}} \\ 0 & 1 & 2\cdot\frac{2v}{2\sqrt{u^2+v^2}} \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{2u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{2v}{\sqrt{u^2+v^2}} \end{vmatrix} = \left\langle -\frac{2u}{\sqrt{u^2+v^2}}, -\frac{2v}{\sqrt{u^2+v^2}}, 1 \right\rangle$$

points upwards ($z > 0$)

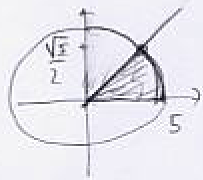
$$\iint \vec{F} \cdot d\vec{S} = \iint_D \left\langle 2v\sqrt{u^2+v^2}, 2u\sqrt{u^2+v^2}, uv \right\rangle \cdot \left\langle -\frac{2u}{\sqrt{u^2+v^2}}, -\frac{2v}{\sqrt{u^2+v^2}}, 1 \right\rangle$$

$$= \iint_D -4uv - 4uv + uv \, dA = \iint_D -7uv \, dA = 0$$ by symmetry

10. (16pts) Sketch the region W given by $x^2+y^2 \le z \le 25$, ~~$x \le y$~~ $x \geq y$, $x, y \geq 0$. Then write the two iterated triple integrals that stand for $\iiint_W f\,dV$ which end in $dz\,dx\,dy$, and $dx\,dz\,dy$.



Projection to xy-plane:

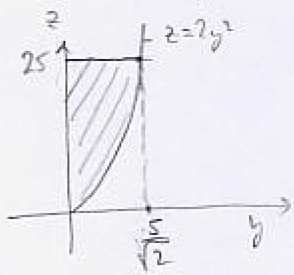


$$\int_0^{\sqrt{5/2}}\int_y^{\sqrt{25-y^2}}\int_{x^2+y^2}^{25} f\,dz\,dx\,dy$$

$x=y \Rightarrow 2x\ 5$

$x=\pm\sqrt{\frac{5}{2}}$

Projection to yz-plane:



$$\int_0^{\frac{5}{\sqrt{2}}}\int_{2y^2}^{25}\int_y^{\sqrt{z-y^2}} f\,dx\,dz\,dy$$

$\begin{cases} z=x^2+y^2 \\ y=x \end{cases}$

$z=2y^2$

Bonus. (15pts) A spherical cap of height h is the set $x^2+y^2+z^2 \leq R^2$, $z \geq R-h$. Show that its surface area is $A = 2\pi Rh$. Then use this formula to get the surface area of a ball or radius R.

See Bonus on Exam 5.