Fall '10/MAT 309/Exam 5 **Name:** Solution *Show all your work.*

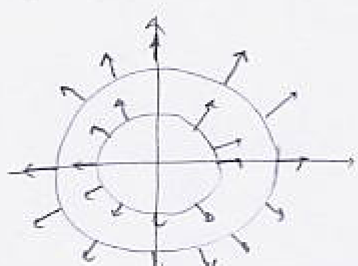
1. (19pts) Let $\phi(x,y) = \sqrt{x^2+y^2}$.
a) Find $\nabla\phi(x,y)$. What is $||\nabla\phi(x,y)||$?
b) Roughly draw the vector field $\nabla\phi(x,y)$, scaling the vectors for a better picture.
c) How could you have roughly done b) without the actual computation in a)?
d) What is $\int_C \nabla\phi \cdot ds$ if C is part of the curve $y = x^3$ from $(0,0)$ to $(2,8)$? How about if C is a straight line segment from $(0,0)$ to $(2,8)$?

a) $\nabla\varphi(x,y) = \left\langle \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x, \frac{1}{2\sqrt{x^2+y^2}}\cdot 2y \right\rangle$

$= \frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$

$\|\nabla\varphi\| = \frac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = 1$

b)



c) $\nabla\varphi$ has vectors that are perpendicular to level curves, which are circles

d) $\int_C \nabla\varphi \cdot ds = \varphi(2,8) - \varphi(0,0)$

$= \sqrt{2^2+8^2} - \sqrt{0} = \sqrt{68} = 2\sqrt{17}$

2. (15pts) Let C be part of the helix $x = 4\cos t$, $y = 4\sin t$, $z = t$, for $t \in [2\pi, 4\pi]$.
a) Set up $\int_C xz\, ds$.
b) Set up $\int_C \mathbf{F}\cdot ds$, if $\mathbf{F}(x,y,z) = \langle y, x, z^2\rangle$.
In both cases simplify the set-up, but do not evaluate the integral.

$\vec{c}\,'(t) = \langle -4\sin t, 4\cos t, 1\rangle \qquad \|\vec{c}\,'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t + 1} = \sqrt{17}$

a) $\int_C xz\, ds = \int_{2\pi}^{4\pi} 4\cos t \cdot t \cdot \sqrt{17}\, dt = \int_{2\pi}^{4\pi} 4\sqrt{17}\, t\cos t\, dt$

b) $\int_C \vec{F}\cdot d\vec{s} = \int_{2\pi}^{4\pi} \langle 4\sin t, 4\cos t, t^2\rangle \cdot \langle -4\sin t, 4\cos t, 1\rangle\, dt$

$= \int_{2\pi}^{4\pi} -16\sin^2 t + 16\cos^2 t + t^2\, dt$

$= \int_{2\pi}^{4\pi} 16(\cos^2 t - \sin^2 t) + t^2\, dt$

3. (16pts) One of the two vectors fields below is not a gradient field, and the other one is (cross partials, remember?). Identify which is which, and find the potential function for the one that is.

$\mathbf{F}(x,y,z) = \langle x^2+y^2, y^2+z^2, z^2+x^2\rangle$

$$\underbrace{\frac{\partial F_1}{\partial y} = 2y \quad \frac{\partial F_2}{\partial x} = 0}$$

different, so $\vec{F}$ cannot be a gradient.

$$\left.\begin{aligned} \frac{\partial G_1}{\partial y} &= 0 = \frac{\partial G_2}{\partial x} \\ \frac{\partial G_1}{\partial z} &= e^z = \frac{\partial G_3}{\partial x} \\ \frac{\partial G_2}{\partial z} &= e^z = \frac{\partial G_3}{\partial y} \end{aligned}\right\}$$

Since G is defined on a simply-connected region, there must be a φ so $\nabla\varphi = \vec{G}$.

$\mathbf{G}(x,y,z) = \langle e^z, e^z, e^z(x+y+z+1)\rangle$

$$\frac{\partial\varphi}{\partial x} = e^z$$

$$\varphi = xe^z + g(y,z)$$

$$e^z = \frac{\partial\varphi}{\partial y} = \frac{\partial g}{\partial y}$$

$$g = ye^z + h(z)$$

$$\varphi = xe^z + ye^z + h(z)$$

$$e^z(x+y+z+1) = \frac{\partial\varphi}{\partial z} = e^z(x+y) + h'(z)$$

so $h'(z) = e^z(z+1)$

$$h(z) = \begin{bmatrix} u = z+1 & dv = e^z \\ du = 1 & v = e^z \end{bmatrix}$$

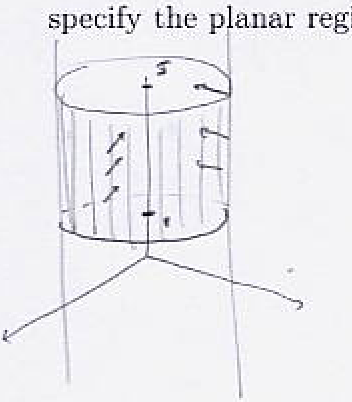
$$= (z+1)e^z - \int e^z\,du$$

$$= (z+1)e^z - e^z$$

$$= ze^z$$

$$\varphi = e^z(x+y+z)$$

4. (24pts) Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$, if S is the part of the cylinder $x^2+y^2=9$ between the planes $z=1$ and $z=5$, and $\mathbf{F}(x,y,z) = \langle -y, x-z, y\rangle$. (The surface does not include the top or the bottom, just part of the cylinder.) Use the normal vectors to the surface that point toward the z-axis. Draw the surface and some normal vectors, parametrize the surface and specify the planar region D where your parameters come from.



$$\vec{T}_\theta \times \vec{T}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos\theta, 3\sin\theta, 0\rangle$$

points away from z-axis, use $-\langle 3\cos\theta, 3\sin\theta, 0\rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \langle -3\sin\theta, 3\cos\theta - z, 3\sin\theta\rangle \cdot \langle -3\cos\theta, -3\sin\theta, 0\rangle\, dA$$

$$= \int_0^{2\pi}\int_1^5 9\sin\theta\cos\theta - 9\sin\theta\cos\theta + 3z\sin\theta\; dz\, d\theta$$

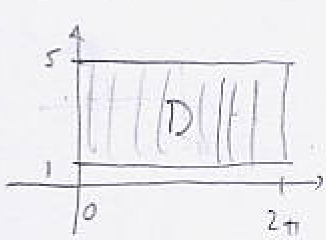
$$= \int_0^{2\pi}\int_1^5 3z\sin\theta\, d\theta = \underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{=0}\cdot\int_1^5 3z\, dz = 0$$

Parametrization:

$x = 3\cos\theta \qquad 0\le\theta\le 2\pi$

$y = 3\sin\theta \qquad 1\le z\le 5$

$z = z$

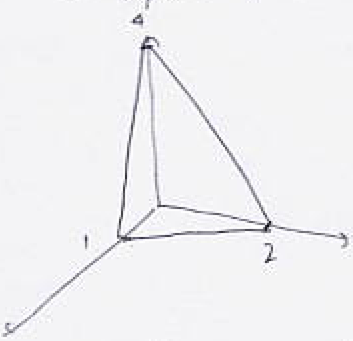


5. (26pts) Find the surface integral $\iint_S xz\,dS$, if S is part of the plane $4x+2y+z=4$ that is in the octant $x, y, z \geq 0$. Draw the surface (intercepts with the axes will help you draw the plane), parametrize it and specify the planar region D where your parameters come from.

$4x+2y+z=4 \quad | \div 4$

$x+\frac{y}{2}+\frac{z}{4}=1$

Intercepts: 1, 2, 4



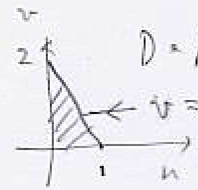
Parametrization:

$x=u$

$y=v$

$z=4-4u-2v$

D = projection to xy-plane



$v=2-2u$

$$\vec{T}_u\times\vec{T}_v=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -4\\ 0 & 1 & -2\end{vmatrix}=\langle 4,2,1\rangle$$

$$\|\langle 4,2,1\rangle\|=\sqrt{16+4+1}=\sqrt{21}$$

$$\iint_S xz\,dS=\iint_D u(4-4u-2v)\sqrt{21}\,dA$$

$$=\sqrt{21}\int_0^1\int_0^{2-2u} 4u(1-u)-2uv\,dv\,du=\sqrt{21}\int_0^1\left(4u(1-u)(2-2u-0)-uv^2\Big|_0^{2-2u}\right)du$$

$$=\sqrt{21}\int_0^1 8u(1-u)^2-u(2-2u)^2\,du=\sqrt{21}\int_0^1 8u(1-u)^2-4u(1-u)^2\,du$$

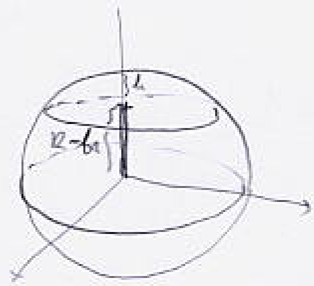
$$=\sqrt{21}\int_0^1 4u(1-u)^2\,du=\left[\begin{matrix}1-u=t & u=1,\ t=0\\ -udu=dt & u=0,\ t=1\end{matrix}\right]=4\sqrt{21}\int_1^0(1-t)t^2(-dt)$$

$$=4\sqrt{21}\int_0^1 t^2-t^3\,dt=4\sqrt{21}\left(\frac{t^3}{3}-\frac{t^4}{4}\right)\Big|_0^1=4\sqrt{21}\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{4}{12}\sqrt{21}=\frac{\sqrt{21}}{3}$$

Area of half a sphere: put $h=R$ in $A=2\pi Rh$

Area of complete sphere $= 2\cdot 2\pi R\cdot R = 4\pi R^2$

Bonus. (10pts) A spherical cap (eek! It again!) of height h is the set $x^2+y^2+z^2 \le R^2$, $z \ge R-h$. Show that its surface area is $A = 2\pi Rh$. Then use this formula to get the surface area of a ball or radius R.



Use spherical coordinates:

$x = R\sin\phi\cos\theta$

$y = R\sin\phi\sin\theta$

$z = R\cos\phi$

$0 \le \theta \le 2\pi$

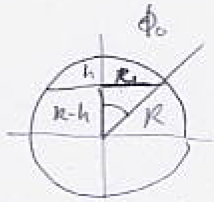
$0 \le \phi \le \phi_0$

$$T_\theta \times T_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \\ R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \end{vmatrix}$$

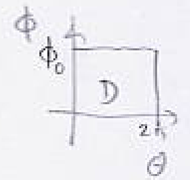
$$= \langle -R^2\sin^2\phi\cos\theta, -R^2\sin^2\phi\sin\theta, -R^2\sin^2\theta\sin\phi\cos\phi - R^2\cos^2\theta\sin\phi\cos\phi \rangle$$

$$= -R^2\sin\phi\langle \sin\phi\cos\theta, \sin\phi\sin\theta, (\sin^2\theta+\cos^2\theta)\cos\phi \rangle$$

$$= -R^2\sin\phi\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi$$



$\cos\phi_0 = \frac{R-h}{R}$



$$\|T_\theta \times T_\phi\| = R^2\sin\phi\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi}$$

$$= R^2\sin\phi\sqrt{\sin^2\phi + \cos^2\phi}$$

$$= R^2\sin\phi$$

$$\text{Area} = \iint_S 1\, dS = \iint_D \|T_u \times T_v\|$$

$$= \int_0^{2\pi}\int_0^{\phi_0} R^2\sin\phi\, d\phi\, d\theta$$

$$= 2\pi R^2\int_0^{\phi_0}\sin\phi\, d\phi = 2\pi r^2(-\cos\phi)\Big|_0^{\phi_0}$$

$$= 2\pi R^2(1-\cos\phi_0) = 2\pi R^2\left(1 - \frac{R-h}{R}\right)$$

$$= 2\pi R^2\cdot\frac{h}{R} = 2\pi Rh.$$

$$\vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -\frac{2r}{2\sqrt{R^2-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \left\langle -\frac{r^2\cos\theta}{\sqrt{R^2-r^2}}, -\frac{r^2\sin\theta}{\sqrt{R^2-r^2}}, r\cos^2\theta + r\sin^2\theta \right\rangle$$

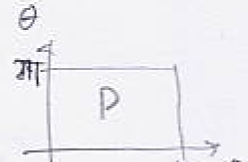
$$= -\frac{r}{\sqrt{R^2-r^2}}\langle r\cos\theta, r\sin\theta, -\sqrt{R^2-r^2} \rangle$$

Cylindrical coordinates:

$x = r\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = r\sin\theta$ $\quad 0 \le r \le R_1$

$z = \sqrt{R^2-r^2}$



$$\|\vec{T}_r \times \vec{T}_\theta\| = \frac{r}{\sqrt{R^2-r^2}}\sqrt{r^2 + R^2 - r^2} = \frac{Rr}{\sqrt{R^2-r^2}}$$

$$\text{Area} = \iint_D \frac{Rr}{\sqrt{R^2-r^2}}\, dA = \int_0^{2\pi}\int_0^{R_1}\frac{Rr}{\sqrt{R^2-r^2}}\, dr\, d\theta$$

$R_1^2 + (R-h)^2 = R^2$

$$= 2\pi R\left(\frac{-1}{2}\right)\frac{2}{1}(R^2-r^2)^{\frac{1}{2}}\Big|_0^{R_1} = -2\pi R\left(\underbrace{(R^2-R_1^2)}_{(R-h)^2}{}^{\frac{1}{2}} - (R^2)^{\frac{1}{2}}\right) = -2\pi R(R-h-R)$$

$$= 2\pi Rh$$