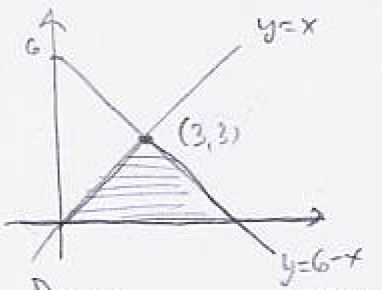
Fall '10/MAT 309/Exam 4 **Name:** Solution *Show all your work.*

1. (18pts) Find $\iint_D y\,dA$ if D is the region bounded by the lines $y = 0$, $y = x$ and $y = 6 - x$. Sketch the region of integration.



Do as horizontally simple.

$x = 6 - x$

$2x = 6$

$x = 3$

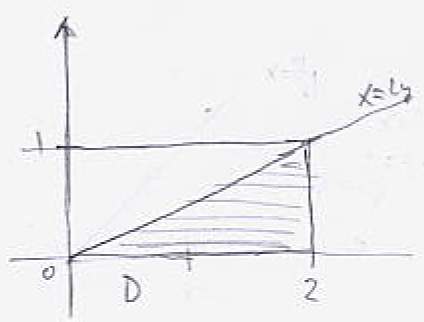
$$\iint_D y\,dA = \int_0^3 \int_y^{6-y} y \, dx\,dy$$

$$= \int_0^3 y(6-y-y)\,dy = \int_0^3 6y - 2y^2\,dy$$

$$= \left(6\frac{y^2}{2} - \frac{2y^3}{3}\right)\Big|_0^3$$

$$= 3\cdot 3^2 - \frac{2\cdot 3^3}{3} = 27 - 18 = 9$$

2. (18pts) Evaluate $\displaystyle\int_0^1 \int_{2y}^{2} ye^{x^3}\,dx\,dy$ by changing the order of integration. Sketch the region of integration.

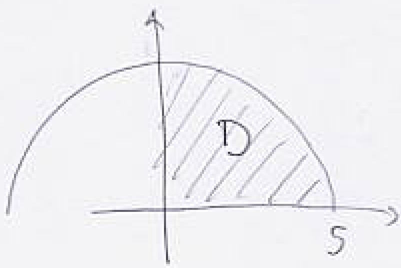


$$\int_0^1 \int_{2y}^{2} ye^{x^3}\,dx\,dy = \int_0^2 \int_0^{\frac{1}{2}x} ye^{x^3}\,dy\,dx$$

$$= \int_0^2 e^{x^3} \frac{1}{2}y^2 \Big|_0^{\frac{1}{2}x} dx = \frac{1}{8}\int_0^2 x^2 e^{x^3}\,dx = \left[\begin{array}{ll} u = x^3 & x=2,\ u=8 \\ du = 3x^2\,dx & x=0,\ u=0 \\ \frac{1}{3}du = x^2\,dx & \end{array}\right]$$

$$= \frac{1}{24}\int_0^8 e^u\,du = \frac{1}{24} e^u \Big|_0^8 = \frac{1}{24}(e^8 - 1)$$

3. (16pts) Use polar coordinates to evaluate the integral $\int_0^5 \int_0^{\sqrt{25-x^2}} (x+y)\, dy\, dx$. Sketch the region of integration first.



$y = \sqrt{25-x^2}$

$y^2 = 25-x^2$

$x^2+y^2=25$

$$\iint_D (x+y)\,dA = \int_0^{\frac{\pi}{2}} \int_0^5 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^5 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^5 r^2\,dr \cdot \int_0^{\frac{\pi}{2}} (\cos\theta+\sin\theta)\,d\theta$$

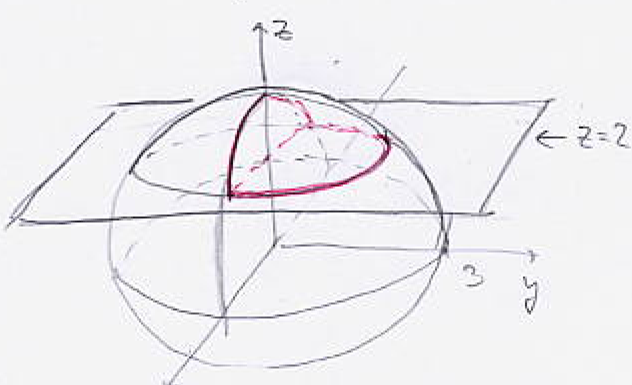
$$= \frac{r^3}{3}\Big|_0^5 \cdot \left(\sin\theta - \cos\theta\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{125}{3}\cdot\left(1-0-(0-1)\right) = \frac{250}{3}$$

4. (16pts) Sketch the region W given by $x^2+y^2+z^2 \le 9$, $z \ge 2$, $y \ge 0$. Then write the two iterated triple integrals that stand for $\iiint_W f\, dV$ which end in $dz\, dy\, dx$, and $dy\, \cancel{dz\, dx}$ $dx\, dz$.

Inside sphere $x^2+y^2+z^2=9$

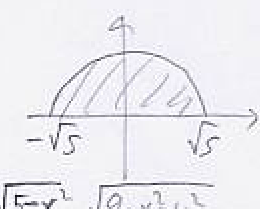
above plane $z=2$, $y \ge 0$



Proj. to xy-plane:

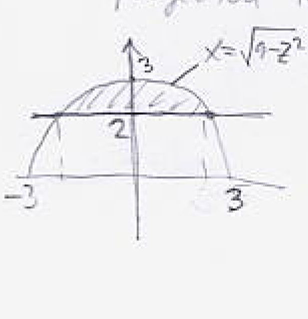
$\begin{cases} x^2+y^2+z^2=9 \\ z=2 \end{cases}$

$x^2+y^2=5$



$$\iiint_W f\,dV = \int_{-\sqrt{5}}^{\sqrt{5}} \int_0^{\sqrt{5-x^2}} \int_2^{\sqrt{9-x^2-y^2}} f\,dz\,dy\,dx$$

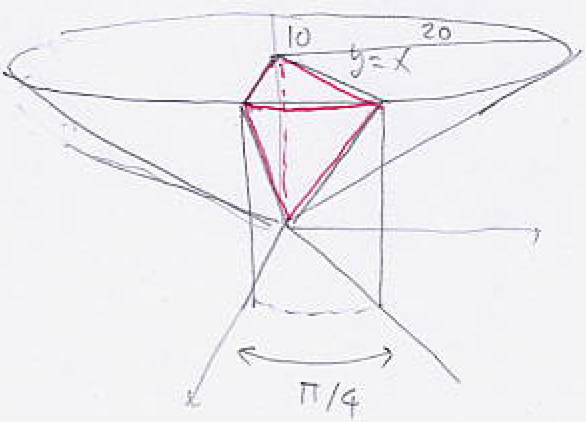
Projection to xz plane:



$$\int_2^3 \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_0^{\sqrt{9-x^2-z^2}} f\,dy\,dx\,dz$$

5. (16pts) Use cylindrical coordinates to set up $\iiint_W \frac{xyz}{x^2+y^2+1}\,dV$ where W is the region above the cone $z = \frac{1}{2}\sqrt{x^2+y^2}$, under the plane $z = 10$ and between the planes $y = x$ and $y = x$ $(x, y \geq 0)$. Sketch the region of integration. Do not evaluate the integral.

$z = \frac{1}{2}\sqrt{x^2+y^2} = \frac{1}{2} r$



$$\iiint_W \frac{xyz}{x^2+y^2+1}\,dV$$

$$= \int_0^{\pi/4}\int_0^{20}\int_{\frac{1}{2}r}^{10} \frac{r\cos\theta\, r\sin\theta\, z}{r^2+1}\, r\,dz\,dr\,d\theta$$

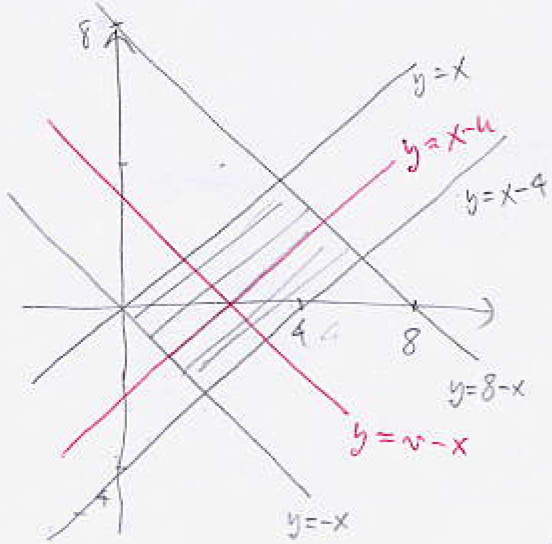
$$= \int_0^{\pi/4}\int_0^{20}\int_{\frac{1}{2}r}^{10} \frac{r^3 z\sin\theta\cos\theta}{r^2+1}\,dz\,dr\,d\theta$$

$z = \frac{1}{2}r$

$z = 10$

$10 = \frac{1}{2}r \qquad r = 20$

6. (16pts) Use change of variables to find the integral $\iint_D e^{x-y}\,dA$ if D is the rectangle bounded by $y = x$, $y = x - 4$, $y = -x$ and $y = 8 - x$. Sketch the region D.



$y - x = u \qquad -4 \leq u \leq -0$

$y + x = v \qquad 0 \leq v \leq 8$

$2y = u + v \qquad y = \frac{1}{2}(u+v)$

$x = v - y = \frac{1}{2}(v-u)$

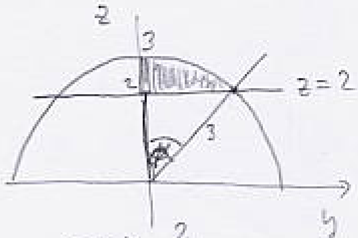
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

$$\iint_D e^{x-y}\,dA = \int_{-4}^{0}\int_0^8 e^{\frac{1}{2}(v-u)-\frac{1}{2}(u+v)}\left|-\frac{1}{2}\right| dv\,du$$

$$= \int_{-4}^{0}\int_0^8 \frac{1}{2}e^{-u}\,dv\,du = 4\cdot\left(-(e^{-u})\Big|_{-4}^{0}\right) = 4(e^4 - 1)$$

Bonus. (10pts) Use spherical coordinates to find the volume of the region W from problem 4.

See picture in problem 4 for region



$(\phi = \alpha)$ $\quad \cos\alpha = \frac{2}{3}$

$\alpha = \arccos\frac{2}{3}$

$z = 2$

$\rho\cos\phi = 2$

$\rho = \frac{2}{\cos\phi}$

$$\int_0^{\pi} \underbrace{\int_0^{\alpha}\int_{\frac{2}{\cos\phi}}^{3} 1\,\rho^2\sin\phi\,d\rho\,d\phi}_{\text{no } \theta \text{ here}}\,d\theta$$

$$\pi \cdot \int_0^{\alpha} \sin\phi\,\frac{1}{3}\rho^3\Big|_{\frac{2}{\cos\phi}}^{3}\,d\phi = \frac{\pi}{3}\int_0^{\alpha}\sin\phi\left(27 - \frac{8}{\cos^3\phi}\right)d\phi$$

$$= \left[\begin{array}{lll} u = \cos\phi & \phi = \alpha = \arccos\frac{2}{3} & \cos\phi = \frac{2}{3} \\ du = -\sin\phi\,d\phi & \phi = 0 & \cos\phi = 1 \end{array}\right]$$

$$= \frac{\pi}{3}\int_1^{\frac{2}{3}}\left(27 - \frac{8}{u^3}\right)(-du) = \frac{\pi}{3}\int_{\frac{2}{3}}^{1}\left(27 - 8u^{-3}\right)du$$

$$= \frac{\pi}{3}\left(27\cdot\left(1 - \frac{2}{3}\right) - 8\,\frac{u^{-2}}{-2}\Big|_{\frac{2}{3}}^{1}\right) = \frac{\pi}{3}\left(9 + 4\underbrace{\left(1 - \frac{9}{4}\right)}_{4-9}\right) = \frac{\pi}{3}\,4 = \frac{4\pi}{3}$$