Spring '10/MAT 309/Exam 3 **Name:** Solution *Show all your work.*

1. (12pts) Find the equation of the tangent plane to the surface $x^2 - \frac{y^2}{4} + z^2 = 1$ at the point $\left(1, \sqrt{2}, \frac{\sqrt{2}}{2}\right)$. Simplify the equation to standard form.

$$\underbrace{x^2 - \frac{y^2}{4} + z^2 - 1}_{F(x,y,z)} = 0$$

Normal vector $= \nabla F$

$$\nabla F = \langle 2x, -\tfrac{2y}{4}, 2z \rangle$$

$$= \langle 2x, -\tfrac{y}{2}, 2z \rangle$$

$$\nabla F\left(1, \sqrt{2}, \tfrac{\sqrt{2}}{2}\right) = \langle 2, -\tfrac{\sqrt{2}}{2}, \sqrt{2} \rangle$$

$$2(x-1) - \frac{\sqrt{2}}{2}(y - \sqrt{2}) + \sqrt{2}\left(z - \frac{\sqrt{2}}{2}\right) = 0$$

$$2x - \frac{\sqrt{2}}{2} y + \sqrt{2} z - 2 + 1 - 1 = 0$$

$$\boxed{2x - \frac{\sqrt{2}}{2} y + \sqrt{2} z = 2}$$

2. (20pts) A bug is moving along the path $\mathbf{r}(t) = \langle 2t+3, t^2 \rangle$ through a region where temperature is distributed according to the function $T(x,y) = \dfrac{e^{x-y}}{x}$ (in °C).

a) Find the point P where the bug is at $t = 3$.

b) At what rate is the bug's temperature changing when $t = 3$ (in seconds)? What are the units?

c) At P, in which direction does the temperature decrease the fastest?

a) $\vec{r}(3) = \langle 9, 9 \rangle$

b) $\frac{d}{dt} F(\vec{r}(t)) = \nabla F \cdot \vec{r}\,'(t)$

$$\nabla F = \left\langle \frac{e^{x-y} \cdot x - e^{x-y} \cdot 1}{x^2}, \frac{1}{x} e^{x-y}(-1) \right\rangle$$

$$= e^{x-y} \left\langle \frac{x-1}{x^2}, -\frac{1}{x} \right\rangle$$

$$\nabla F(9,9) = e^0 \left\langle \frac{8}{81}, -\frac{1}{9} \right\rangle = \left\langle \frac{8}{81}, -\frac{1}{9} \right\rangle$$

$$\vec{r}\,'(t) = \langle 2, 2t \rangle$$

$$\vec{r}\,'(3) = \langle 2, 6 \rangle$$

$$\nabla F(9,9) \cdot \vec{r}\,'(3)$$

$$= \left\langle \frac{8}{81}, -\frac{1}{9} \right\rangle \cdot \langle 2, 6 \rangle$$

$$= \frac{16}{81} - \frac{6}{9} = \frac{16 - 54}{81} = -\frac{38}{81} \ {}^\circ C/s$$

c) In the direction of $-\nabla F(3,3)$, which is $\left\langle -\frac{8}{81}, \frac{1}{9} \right\rangle$

3. (20pts) Let $f(x,y) = x\ln(x^2+y^2)$, $x = \sin u + \cos v$, $y = \cos u \sin v$. Find $\dfrac{\partial f}{\partial u}$ when $u = \pi$, $v = \frac{\pi}{2}$.

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \left(\ln(x^2+y^2) + 1\cdot\frac{1}{x^2+y^2}\cdot 2x\right)\cdot\cos u +$$

$$+ x\cdot\frac{1}{x^2+y^2}\cdot 2y\cdot(-\sin u\sin v)$$

When $u=\pi$, $v=\frac{\pi}{2}$: $x = 0+0=0$, $y = -1\cdot 1 = -1$

$$= \left(\ln(x^2+y^2) + \frac{2x}{x^2+y^2}\right)\cos u - \frac{2xy}{x^2+y^2}\sin u \sin v$$

$$\left.\frac{\partial f}{\partial u}\right|_{\substack{u=\pi\\ v=\frac{\pi}{2}}} = \left(\ln 1 + \frac{0}{0+1}\right)\cdot(-1) - \frac{2\cdot 0\cdot(-1)}{0+1}\cdot 0\cdot 1 = -\ln 1 = 0$$

4. (12pts) A cylinder is measured to have radius $x = 20$cm and height $y = 12$cm, with an error in measurement at most 0.75cm in each. Estimate the maximal error in computing the volume of the cylinder.

$V = \pi x^2 y$

$$\Delta V \approx \frac{\partial V}{\partial x}\cdot\Delta x + \frac{\partial V}{\partial y}\cdot\Delta y$$

$$= 2\pi x y\,\Delta x + \pi x^2\cdot\Delta y$$

Putting in
$x=20$
$y=12$
$\Delta x = \frac{3}{4}$
$\Delta y = \frac{3}{4}$

Putting in $x=20$, $y=12$, $\Delta x = \Delta y = \frac{3}{4}$

$$\Delta V \approx 2\pi\cdot \overset{5}{\cancel{20}}\cdot 12\cdot\frac{3}{\cancel{4}} + \pi\cdot 20^2\cdot\frac{3}{4}$$

$$= 360\pi + 300\pi$$

$$= 660\pi \text{ cm}^3$$

estimated maximal error

5. (12pts) Find $\frac{\partial y}{\partial z}$ using implicit differentiation, if $x^2y + y^3z + z^4x = 13$.

$$\underbrace{x^2y + y^3z + z^4x - 13 = 0}_{F(x,y,z)}$$

$$\frac{\partial y}{\partial z} = -\frac{F_z}{F_y} = -\frac{y^3 + 4z^3x}{x^2 + 3y^2z}$$

6. (24pts) Find and classify the local extremes for the function $f(x,y) = x^2y^2 + y^4 - x^2 + 6y$.

$f_x = 2xy^2 - 2x$

$f_y = 2x^2y + 4y^3 + 6$

$$D = \begin{vmatrix} 2y^2 - 2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{vmatrix}$$

$$\begin{cases} 2x(y^2-1) = 0 \\ 2x^2y + 4y^3 + 6 = 0 \end{cases}$$

First equation gives:

$x = 0$ or $y = 1$ or $y = -1$

put in second:

$x=0$	$y=1$	$y=-1$
$4y^3 + 6 = 0$	$2x^2 + 10 = 0$	$-2x^2 - 4 + 6 = 0$
$y^3 = -\frac{3}{2}$	$x^2 = -5$	$-2x^2 + 2 = 0$
$y = \sqrt[3]{-\frac{3}{2}}$	no solution	$x^2 = 1$
		$x = \pm 1$

Candidates: $\left(0, \sqrt[3]{-\frac{3}{2}}\right), (1,-1), (-1,-1)$

$$D\left(0, \sqrt[3]{-\tfrac{3}{2}}\right) = \begin{vmatrix} \overbrace{2\left(\left(\sqrt[3]{-\frac{3}{2}}\right)^2 - 1\right)}^{>0 \text{ since } \sqrt[3]{\frac{3}{2}} > 1} & 0 \\ 0 & \underbrace{12 \cdot \left(\sqrt[3]{-\frac{3}{2}}\right)^2}_{>0} \end{vmatrix} > 0$$

local min at $\left(0, \sqrt[3]{-\frac{3}{2}}\right)$

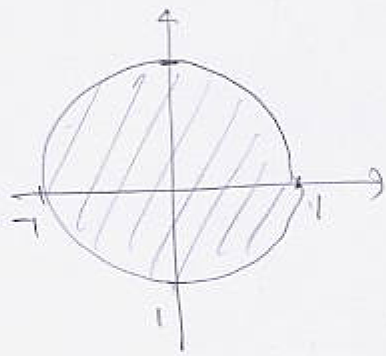
$$D(1,-1) = \begin{vmatrix} 0 & -4 \\ -4 & 14 \end{vmatrix} = -16 < 0$$

$$D(-1,-1) = \begin{vmatrix} 0 & 4 \\ 4 & 14 \end{vmatrix} = -16$$

saddle points at $(1,-1)$ and $(-1,-1)$

Bonus (10pts) Consider the function $f(x, y) = x + \sqrt{3}y$ on the domain $x^2 + y^2 \leq 1$.
a) Determine the global minimum and maximum values of f.
b) Draw the graph of f and justify your answer from a) using the picture.

$x^2+y^2 \leq 1$ is a unit disk



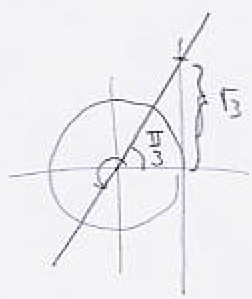
a) 1) critical points:

$f_x = 1$, $f_y = \sqrt{3}$ } never equal 0, so none

2) boundary $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

$g(t) = f(\cos t, \sin t) = \cos t + \sqrt{3}\sin t$

$g'(t) = -\sin t + \sqrt{3}\cos t$



$\sqrt{3}\cos t - \sin t = 0$

$\sqrt{3}\cos t = \sin t \quad | \div \cos t$

$\tan t = \sqrt{3}$

$t = \frac{\pi}{3}, \frac{4\pi}{3}$ critical

$t = 0, 2\pi$ endpoints.

contains origin
$\vec{n} = \langle 1, \sqrt{3}, -1 \rangle$,
intersects xy-plane in
$x + \sqrt{3}y = 0$, $y = -\frac{1}{\sqrt{3}}x$

	Candidates	$f(x,y)$
$t=\frac{\pi}{3}$	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$\frac{1}{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = \frac{4}{2} = 2$ abs max
$t=\frac{4\pi}{3}$	$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$	$-\frac{1}{2} - \sqrt{3}\cdot\frac{\sqrt{3}}{2} = -2$ abs min.
$t=0, 2\pi$	$(1, 0)$	1

b) Graph of $z = x + \sqrt{3}y$ is a plane

Consider the part above the unit disk