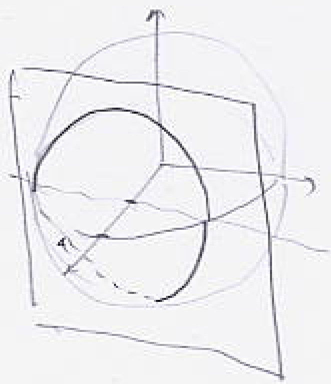
Fall '10/MAT 309/Exam 2 Name: Solution *Show all your work.*

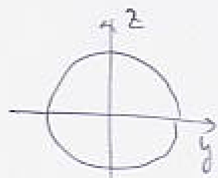
1. (10pts) Write the parametrization of the circle that is the intersection of the sphere $x^2 + y^2 + z^2 = 16$ with the plane $x = 2$. Sketch a picture.



Put $x=2$ into equation:

$4+y^2+z^2=16$

$y^2+z^2=12$



Circle is of radius $\sqrt{12}=2\sqrt{3}$

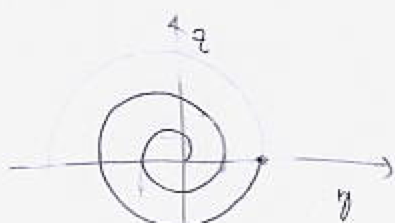
Parametrization:

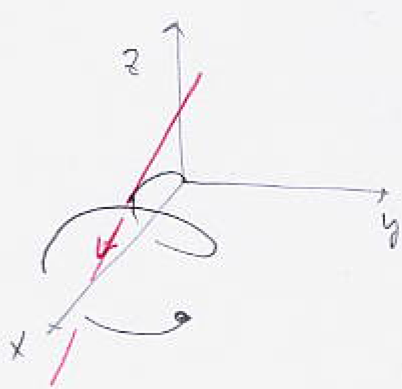
$\vec{r}(t)=\langle 2, 2\sqrt{3}\cos t, 2\sqrt{3}\sin t\rangle$

$t\in[0,2\pi]$

2. (20pts) A curve is given by $\mathbf{r}(t) = \langle 4t, t\cos t, t\sin t\rangle$, $t \in [0, 4\pi]$.
a) Sketch this curve.
b) Find the parametric equation of the tangent line to the curve at time $t = \pi$ and draw this tangent line on your sketch.

a) Projection to yz-plane is a spiral: (2 loops)





b) $\vec{r}'(t)=\langle 4, \cos t - t\sin t, \sin t + t\cos t\rangle$

$\vec{r}(\pi)=\langle 4\pi, -\pi, 0\rangle$

$\vec{r}'(\pi)=\langle 4, -1, -\pi\rangle$

Tangent line:

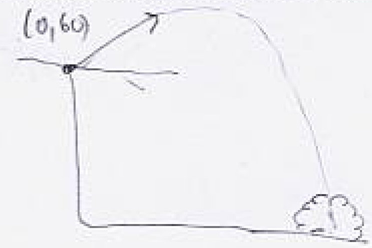
$x = 4\pi + 4t$

$y = -\pi - t$

$z = -\pi t$

3. (22pts) After another ill-fated attempt at lunch, Wile E. Coyote finds himself ejected from the edge of a 60-meter deep canyon at angle $30°$ above the horizontal with initial speed 40 meters per second.
a) Find his position at time t. (For simplicity of calculation, blaspheme away and set $g = 10$.)
b) When does he hit the bottom of the canyon?
c) What is his speed when he hits the bottom?

a)



$\vec{a}(t) = \langle 0, -10\rangle$

$\vec{v}(0) = \langle 40\cos 30°, 40\sin 30°\rangle$

$= \langle 20\sqrt{3}, 20\rangle$

$\vec{r}(0) = \langle 0, 60\rangle$

$\vec{v}(t) = \langle 0, -10t\rangle + \vec{C}$

$\langle 20\sqrt{3}, 20\rangle = \vec{v}(0) = \vec{C}$

$\vec{v}(t) = \langle 20\sqrt{3}, 20 - 10t\rangle$

$\vec{r}(t) = \langle 20\sqrt{3}t, 20t - 10\frac{t^2}{2}\rangle + \vec{D}$

$\langle 0, 60\rangle = \vec{r}(0) = \langle 0, 0\rangle + \vec{D}$

$\vec{r}(t) = \langle 20\sqrt{3}t, -5t^2 + 20t + 60\rangle$

b) y-coord needs to be 0:

$-5t^2 + 20t + 60 = 0 \quad | \div -5$

$t^2 - 4t - 12 = 0$

$(t-6)(t+2) = 0$

$t = 6$ ← hits bottom at time $t=6$

c) $\vec{v}(6) = \langle 20\sqrt{3}, -40\rangle$

Speed $= \|\vec{v}(6)\| = \sqrt{(20\sqrt{3})^2 + (-40)^2}$

$= \sqrt{1200 + 1600} = \sqrt{\underset{400\cdot 7}{2800}} = 20\sqrt{7}$ m/s

4. (18pts) Find the length of the curve with the parametrization $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, \frac{2\sqrt{2}}{\sqrt{3}} t^{\frac{3}{2}}, 3t + 7 \right\rangle$, $t \in [1, 5]$.

$L = \int_1^5 \|\vec{r}'(t)\|\,dt$

$\vec{r}'(t) = \langle \frac{2t}{2}, \frac{2\sqrt{2}}{\sqrt{3}}\frac{3}{2}t^{\frac{1}{2}}, 3\rangle$

$= \langle t, \sqrt{6}\,t^{\frac{1}{2}}, 3\rangle$

$\|\vec{r}'(t)\| = \sqrt{t^2 + 6t + 9}$

$= \sqrt{(t+3)^2} = t + 3$

$\int_1^5 (t+3)\,dt = \left(\frac{t^2}{2} + 3t\right)\Big|_1^5$

$= \frac{1}{2}(25 - 1) + 3(5 - 1)$

$= 12 + 12 = 24$ units

5. (20pts) Let $f(x,y) = x^2y$.

a) Identify and draw vertical traces for this function.

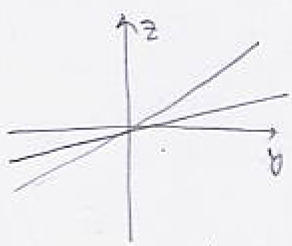
b) Using a), draw the graph of the function (in your 3-D coordinate system, orient the x-axis to the right, and the y-axis away from you).

c) Draw a rough contour map for the function, with contour interval 1, going from $c = -3$ to $c = 3$.

d) By looking at the contour map, indicate the direction (if any), in which we would have to move from $(1,2)$ in order to decrease the value of the function.

a) Vertical traces: $z = yx^2$

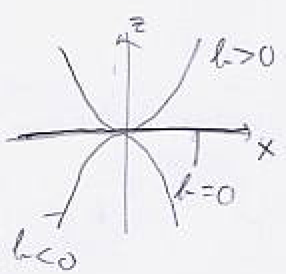
$x = a$ $\quad z = a^2y$



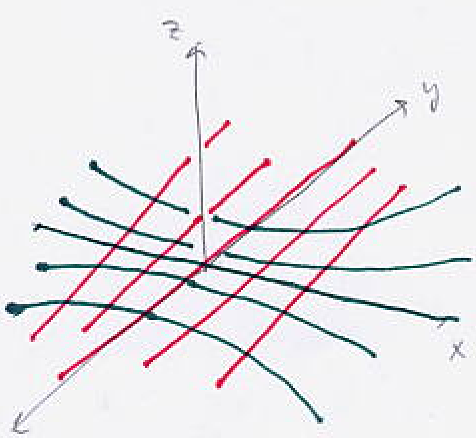
lines with slope ≥ 0

$y = b$ $\quad z = bx^2$

parabolas and a line

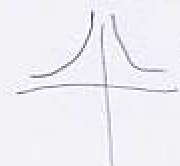


b)

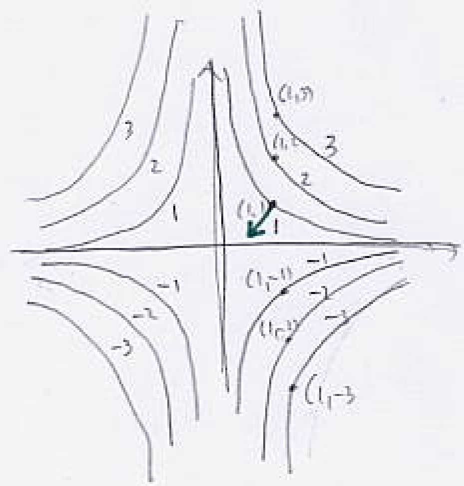


c) $x^2y = c$

$y = \frac{c}{x^2}$



$y = \frac{1}{x^2}$



d) Move towards a contour line with a lower c, say, toward the origin.

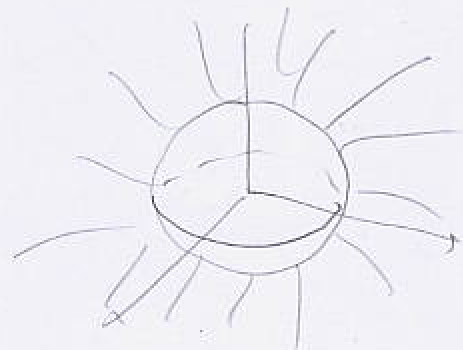
6. (10pts) Determine and sketch the domain of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 9}$.

Must have: $x^2+y^2+z^2-9 \geq 0$

$\underbrace{x^2+y^2+z^2}_{} \geq 9$

$\text{distance to origin}^2 \geq 9$

so $\text{distance to origin} \geq 3$



outside of ball of radius 3, centered at origin

Bonus (10pts) Let $\mathbf{r}(t)$ the position of a moving object in space. If $\mathbf{r}'''(t) = \mathbf{0}$, use differentiation rules for products to help you show that the volume of the parallelepiped spanned by the position, velocity and acceleration vectors is constant. *(Hint: triple product.)*

$V(t) = \vec{r}(t) \cdot (\vec{v}(t) \times \vec{a}(t))$

$$\frac{dV}{dt} = \vec{r}\,'(t) \cdot (\vec{v}(t) \times \vec{a}(t)) + \vec{r}(t) \cdot (\vec{v}(t) \times \vec{a}(t))'$$

$$= \vec{v}(t) \cdot (\vec{v}(t) \times \vec{a}(t)) + \vec{r}(t) \cdot (\vec{v}\,'(t) \times \vec{a}(t) + \vec{v}(t) \times \vec{a}\,'(t))$$

$$= \underbrace{\vec{v}(t) \cdot (\vec{v}(t) \times \vec{a}(t))}_{=0} + \vec{r}(t) \cdot (\underbrace{\vec{a}(t) \times \vec{a}(t)}_{=\vec{0}}) + \vec{r}(t) \cdot (\vec{v}(t) \times \underbrace{\vec{r}\,'''(t)}_{=\vec{0}}) = 0$$

because $\vec{v}(t) \times \vec{a}(t)$ is perpendicular to $\vec{v}(t)$

since $\vec{a}(t)$ and $\vec{a}(t)$ are parallel

by assumption

Since $\frac{dV}{dt} = 0$, we conclude $V(t) =$ constant.