Fall '10/MAT 309/Exam 1 **Name:** Solution *Show all your work.*

1. (~~20~~ 22pts) Let $\mathbf{u} = \langle 1, 2, -3 \rangle$ and $\mathbf{v} = \langle 4, -1, -4 \rangle$.
a) Calculate $3\mathbf{u}$, $2\mathbf{u} - 4\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$ and $||v||$.
b) Find the unit vector in direction of $\mathbf{v}$.
c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

a) $3\vec{u} = \langle 3, 6, -9 \rangle$

$2\vec{u} - 4\vec{v} = \langle 2, 4, -6 \rangle - \langle 16, -4, -16 \rangle$

$= \langle -14, 8, 10 \rangle$

$\vec{u} \cdot \vec{v} = 1 \cdot 4 + 2 \cdot (-1) + (-3) \cdot (-4)$

$= 4 - 2 + 12 = 14$

$||\vec{v}|| = \sqrt{16 + 1 + 16} = \sqrt{33}$

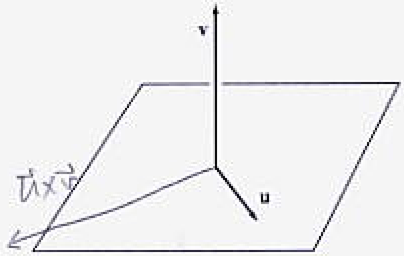
b) $\vec{e}_v = \frac{1}{\sqrt{33}} \langle 4, -1, -4 \rangle$

c) $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||} = \frac{14}{\sqrt{1+4+9} \cdot \sqrt{33}}$

$= \frac{14}{\sqrt{14}\sqrt{33}} = \sqrt{\frac{14}{33}}$

$\theta = \arccos\sqrt{\frac{14}{33}}$

2. (8pts) Vectors $\mathbf{u}$ and $\mathbf{v}$ are drawn below (they are perpendicular). Their lengths are $||\mathbf{u}|| = 3$ and $||\mathbf{v}|| = 1.5$. Draw the vector $\mathbf{u} \times \mathbf{v}$ and state its length.



$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin\frac{\pi}{2}$

$= 3 \cdot 1.5 = 4.5$

3. (12pts) Find the point of intersection of the line $x = 2 + t$, $y = -3 + 2t$, $z = 5t$ with the plane $2x - 3y + z = 11$.

$2(2+t) - 3(-3+2t) + 5t = 11$

$4 + 2t + 9 - 6t + 5t = 11$

$t = -2$

The point is

$(2+(-2), -3+2(-2), 5\cdot(-2))$

$= (0, -7, -10)$

4. (20pts) Two lines are given parametrically: $x = 1 - t$, $y = 4 + 2t$, $z = 3 + 2t$ and $x = 2t$, $y = 1 - 4t$, $z = -3 - 4t$.
a) Show that these lines are parallel.
b) Find the equation of the plane spanned by these two lines.

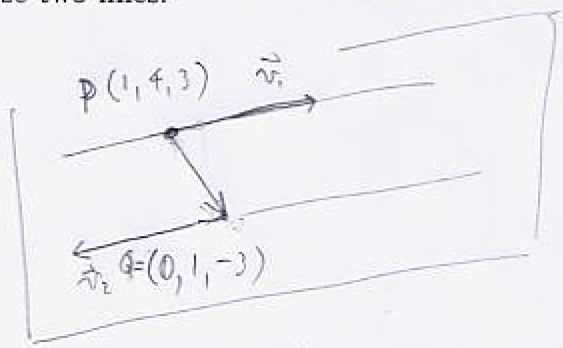
$x = 1 - t \quad x = 2t$

$y = 4 + 2t \quad y = 1 - 4t$

$z = 3 + 2t \quad z = -3 - 4t$

$\vec{v}_1 = \langle -1, 2, 2 \rangle \quad \vec{v}_2 = \langle 2, -4, -4 \rangle$

Since direction vectors are multiples of each other ($\vec{v}_2 = -2\vec{v}_1$) they are parallel,



Use $\vec{v}_1 \times \vec{PQ}$ to get the normal vector

$\vec{PQ} = \langle -1, -3, -6 \rangle$ may use $\vec{QP}$.

$$\vec{v}_1 \times \vec{QP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 2 \\ 1 & 3 & 6 \end{vmatrix} = (12-6)\vec{i} - (-6-2)\vec{j} + (-3-2)\vec{k}$$

$= 6\vec{i} + 8\vec{j} - 5\vec{k}$

Equation of plane:

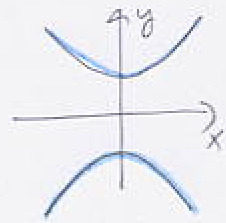
$6x + 8y - 5z = 6\cdot 1 + 8\cdot 4 - 5\cdot 3$

$6x + 8y - 5z = 23$

5. (16pts) This problem is about the surface $-\left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2-\left(\frac{z}{3}\right)^2=1$.
a) Identify and sketch the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

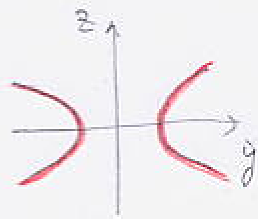
a) $z=0$ $\quad \left(\frac{y}{4}\right)^2-\left(\frac{x}{3}\right)^2=1$

hyperbola

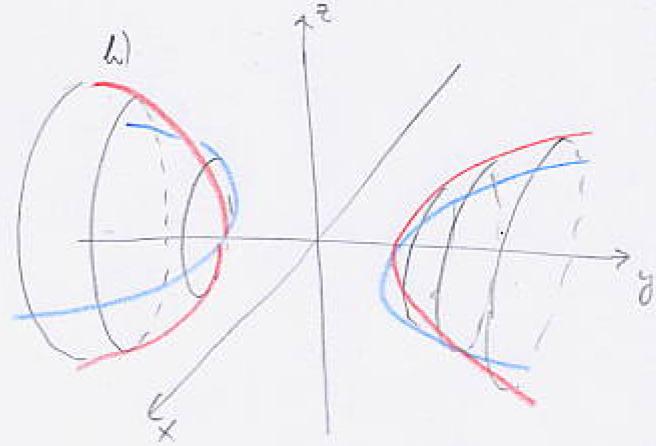


$y=0$ $\quad \underbrace{-\left(\frac{x}{3}\right)^2-\left(\frac{z}{3}\right)^2}_{\le 0}=1$ $\quad$ nothing

$x=0$ $\quad \left(\frac{y}{4}\right)^2-\left(\frac{z}{3}\right)^2=1$

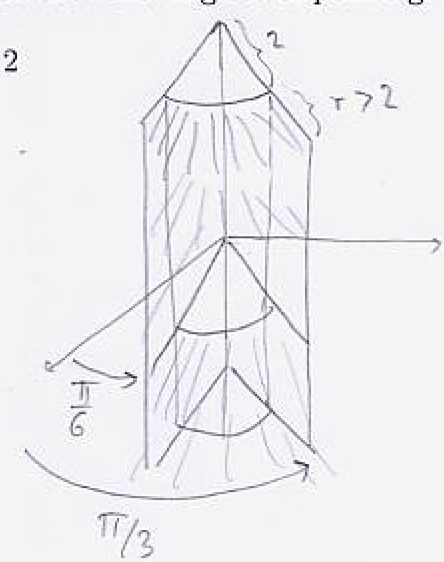


b)



Hyperboloid of two sheets

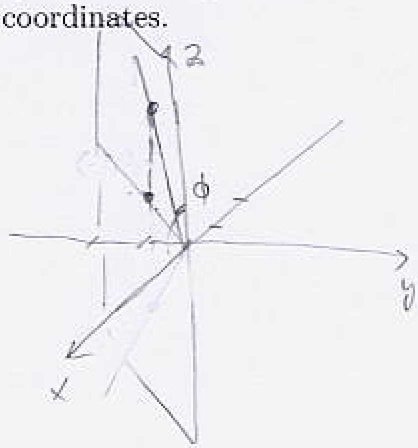
6. (10pts) Sketch the following set of points given in cylindrical coordinates:
$\frac{\pi}{6}\le\theta\le\frac{\pi}{3}$, $r>2$



Part of a wedge,
<u>outside</u> the cylinder $r=2$

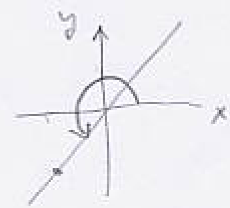
7. (12pts) Sketch the point whose rectangular coordinates are $(-2, -2, \sqrt{\frac{8}{3}})$ and find its spherical coordinates.

≈ 3



$$\rho = \sqrt{4+4+\frac{8}{3}} = \sqrt{\frac{32}{3}} = \frac{4\sqrt{2}}{3}$$

$$\tan\theta = \frac{-2}{-2} = 1$$



$\theta = \frac{5\pi}{4}$, because (x,y) are in 3rd quadrant

$$(\rho, \theta, \phi) = \left(\frac{4\sqrt{2}}{3}, \frac{5\pi}{4}, \frac{\pi}{3}\right)$$

$$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{\frac{8}{3}}}{\sqrt{\frac{32}{3}}} = \sqrt{\frac{8}{\cancel{3}} \cdot \frac{\cancel{3}}{32}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}$$

Bonus (10pts) Refer to the parallel lines of problem 4.

a) Show that the lines are not identical. *(Hint: show a point on one line is not on the other.)*

b) Find the distance between those lines. *(Hints: one way uses the area of a parallelogram. Another uses a plane perpendicular to the lines.)*

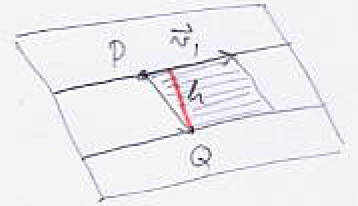
a) $(1,4,3)$ is a point on the first line.

See if it is on the second one:

$$\left.\begin{array}{ll} 1 = 2t & t = \frac{1}{2} \\ 4 = 1-4t & t = -\frac{3}{4} \\ 3 = -3-4t & \end{array}\right\} \text{ no solution}$$

We cannot satisfy these three equations with a single t, so $(1,4,3)$ is not on the second line.

b)



(Refer to picture and computation in problem 4)

$$\text{distance} = h = \frac{\text{area of parallelogram}}{\|v_1\|} = \frac{\|\vec{v}_1 \times \vec{PQ}\|}{\|\vec{v}_1\|} = \frac{\sqrt{36+64+25}}{\sqrt{1+4+4}} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$$

Other way: Plane through P with normal vector $\vec{v}_1 = \langle -1, 2, 2\rangle$:

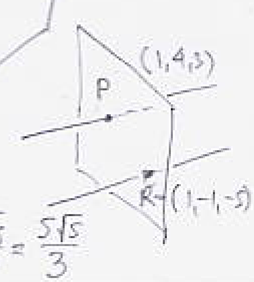
$$-x + 2y + 2z = -1 + 8 + 6$$

$$x - 2y - 2z = -13$$

Intersection with second line: $1 + \frac{34}{9}$ $-3 + \frac{24}{9}$

$$2t - 2(1-4t) - 2(-3-4t) = 13 \qquad R = \left(2\cdot\left(-\tfrac{17}{18}\right), 1 - 4\left(-\tfrac{17}{18}\right), -3-4\left(-\tfrac{17}{18}\right)\right)$$

$$18t + 4 = -13, \quad t = -\frac{17}{18}, \quad R = \left(-\frac{17}{9}, \frac{43}{9}, \frac{7}{9}\right)$$



$$d(P,R) = \sqrt{\left(-\tfrac{17}{9}-1\right)^2 + \left(\tfrac{43}{9}-4\right)^2 + \left(\tfrac{7}{9}-3\right)^2}$$

$$= \sqrt{\left(\tfrac{26}{9}\right)^2 + \left(\tfrac{7}{9}\right)^2 + \left(\tfrac{20}{9}\right)^2} = \sqrt{\frac{676+49+400}{81}} = \sqrt{\frac{1125}{81}} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$$