Spring '08/MAT 250/Final Exam **Name:** Solution *Show all your work.*

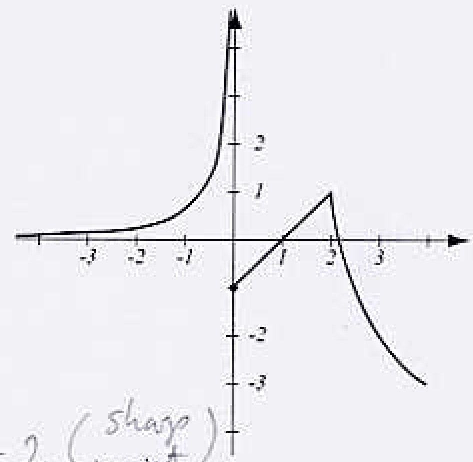
1. (14pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x\to 0^-} f(x) = \infty$

$\lim_{x\to 0^+} f(x) = -1$

$\lim_{x\to 0} f(x) =$ d.n.e. (one-sided limits are different)

$\lim_{x\to -\infty} f(x) = 0$



List points where f is not continuous and explain why.

At $x=0$, $\lim_{x\to 0} f(x)$ d.n.e.

List points where f is not differentiable and explain why.

At $x=0$ (not continuous there) and $x=2$ (sharp point)

2. (12pts) Find $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ in two ways: a) algebraically b) Using L'Hospital's rule

a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$

b) $\lim_{x\to 4} \frac{\overset{\to 0}{\sqrt{x}-2}}{\underset{\to 0}{x-4}} \overset{\text{L'H}}{=} \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

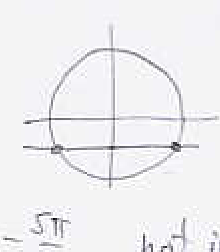
3. (10pts) Find the absolute minimum and maximum values for the function $f(x) = x - 2\cos x$ on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$f'(x) = 1 + 2\sin x$

critical pts:

$1 + 2\sin x = 0$

$\sin x = -\frac{1}{2}$



$x = -\frac{\pi}{6}, -\frac{5\pi}{6}$ not in interval

x	$x - 2\cos x$	
$-\frac{\pi}{6}$	$-\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{2} = -2.25$	abs min
$-\frac{\pi}{2}$	$-\frac{\pi}{2} \approx -1.57$	
$\frac{\pi}{2}$	$\frac{\pi}{2} \approx 1.57$	abs. max.

4. (10pts) Use implicit differentiation to find y'.

$$y\cos x = \frac{x}{y} \quad \Big| \frac{d}{dx}$$

$$y'\cos x + y(-\sin x) = \frac{1\cdot y - x\cdot y'}{y^2} \quad \Big| \cdot y^2$$

$$y'y^2\cos x - y^3\sin x = y - xy'$$

$$y'y^2\cos x + xy' = y + y^3\sin x$$

$$y'(y^2\cos x + x) = y + y^3\sin x$$

$$y' = \frac{y + y^3\sin x}{y^2\cos x + x}$$

5. (14pts) Let $V(t) = t^3 - 5t^2 + 4t + 7$ be the volume, measured in liters, of water in a tank at time t minutes.
a) What is the volume at $t = 2$?
b) Find $V'(2)$. What does it represent? What are the units?
c) Use the numbers from a) and b) to approximate the volume at time $t = 2.2$.
d) What is the exact volume at time $t = 2.2$?

a) $V(2) = 8 - 20 + 8 + 7 = 3\,\ell$

b) $V'(t) = 3t^2 - 10t + 4$

$V'(2) = 12 - 20 + 4 = -4\,\ell/\text{min}$

At time $t = 2$ water is draining at rate $4\,\ell/\text{min}$

c) $V(2.2) \approx V(2) + 0.2\cdot V'(2)$

$= 3 + 0.2(-4)$

$= 2.2$

d) $V(2.2) = 2.248\,\ell$

6. (21pts) Let $f(x) = x^2 e^x$.

a) Find the horizontal asymptotes.

b) Find the intervals of increase/decrease and where f has a local maximum and minimum.

c) Find the intervals where f is concave up or down and where it has an inflection point.

d) Use your calculator and the results of a), b) and c) to accurately sketch the graph of f.

a) $\lim_{x\to\infty} x^2 e^x = \infty \cdot \infty = \infty$

$$\lim_{x\to-\infty} x^2 e^x = \lim_{x\to-\infty} \frac{x^2}{e^{-x}} \overset{L'H}{=} \lim_{x\to-\infty} \frac{2x}{-e^{-x}} \overset{L'H}{=} \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{\infty} = 0$$

$\infty \cdot 0$, indet. form

$$f'(x) = 2xe^x + x^2 e^x = e^x(x^2+2x)$$

always defined

$$f''(x) = e^x(x^2+2x) + e^x(2x+2) = e^x(x^2+4x+2)$$

always defined

Critical pts: $e^x(x^2+2x) = 0$, $e^x > 0$

$x^2+2x=0$

$x(x+2)=0$

$x = 0, -2$

Sign of f' depends only on x^2+2x

$e^x(x^2+4x+2)=0$

$x^2+4x+2=0$

$$x = \frac{-4 \pm \sqrt{16-8}}{2} = \frac{-4 \pm \sqrt{8}}{2} = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

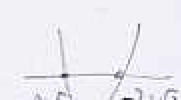
$= -3.41, -0.59$

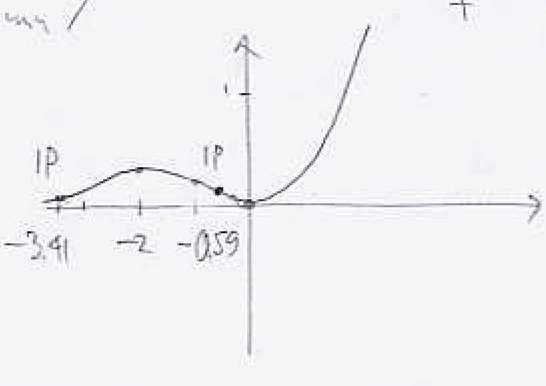
Sign of f'' only depends on x^2+4x+2

x^2+2x

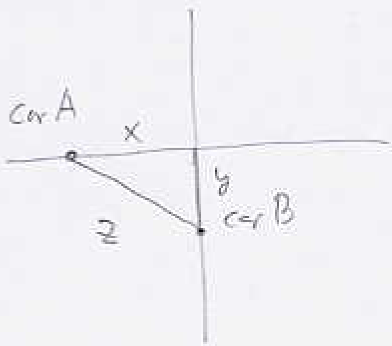
		-2		0	
x^2+2x	$+$	0	$-$	0	$+$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	loc. max	$\searrow$	loc. min	$\nearrow$

		$-2-\sqrt{2}$		$-2+\sqrt{2}$	
x^2+4x+2	$+$	0	$-$	0	$+$
f	CU	IP	CD	IP	CU





7. (13pts) At time $t = 0$ a car starts moving westward from an intersection at speed 45mph. At time $t = 1$hr another car starts moving southward from the same intersection, with speed 60mph. At what rate are the cars moving apart at time $t = 2$hrs?



Know $x' = 45$

$y' = 60$

Need $\frac{dz}{dt}$ when $t=2$

$x^2 + y^2 = z^2 \quad \Big| \frac{d}{dt}$

$2xx' + 2yy' = 2zz'$

$z' = \frac{xx' + yy'}{z}$

When $t=2$

$x = 2 \cdot 45 = 90$

$y = 1 \cdot 60 = 60$

$z^2 = 90^2 + 60^2$

$z^2 = 14500$

$z = 10\sqrt{145}$

$z' = \frac{90 \cdot 45 + 60 \cdot 60}{10 \cdot \sqrt{145}} = \frac{9 \cdot 45 + 6 \cdot 60}{\sqrt{145}}$

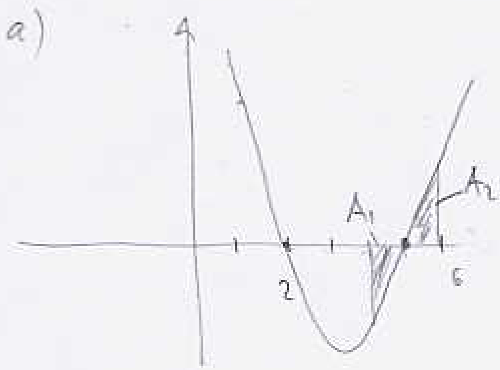
$= \frac{765}{\sqrt{145}} = 63.529792$ mph

8. (12pts) Consider the integral $\int_4^6 x^2 - 7x + 10\,dx$.

a) Use a picture to determine whether this definite integral is positive or negative.

b) Evaluate the integral and verify your conclusion from a).

a)



$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$\int_4^6 x^2 - 7x + 10\,dx = -A_1 + A_2 > 0$

since A_2 appears slightly bigger.

b) $\int_4^6 x^2 - 7x + 10\,dx$

$= \left(\frac{x^3}{3} - 7\frac{x^2}{2} + 10x\right)\Big|_4^6$

$= \frac{1}{3}(6^3 - 4^3) - \frac{7}{2}(6^2 - 4^2) + 10(6-4)$

$= \frac{1}{3} 152 - \frac{7}{2} 20 + 20$

$= \frac{152}{3} - 70 + 20 = \frac{152}{3} - 50 = \frac{2}{3}$

9. (6pts) Find $f(x)$ if $f'(x) = \dfrac{1}{x^5}$ and $f(1) = 3$.

$$f'(x) = x^{-5}$$

$$f(x) = \frac{x^{-4}}{-4} + C$$

$$= -\frac{1}{4x^4} + C$$

$$3 = f(1) = -\frac{1}{4 \cdot 1} + C$$

$$C = 3 + \frac{1}{4} = \frac{13}{4}$$

$$f(x) = -\frac{1}{4x^4} + \frac{13}{4}$$

10. (6pts) Find the indefinite integral:

$$\int \sqrt{x} - \sec^2 x \, dx = \int x^{\frac{1}{2}} - \sec^2 x \, dx$$

$$= \frac{x^{3/2}}{\frac{3}{2}} - \tan x$$

$$= \frac{2}{3} x^{3/2} - \tan x + C$$

11. (10pts) Use substitution (don't forget to change bounds) to evaluate:

$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos^4 x \sin x \, dx = \begin{bmatrix} u = \cos x & x = \frac{2\pi}{3},\ u = -\frac{1}{2} \\ du = -\sin x \, dx & x = \frac{\pi}{2},\ u = 0 \end{bmatrix}$$

$$= \int_0^{-\frac{1}{2}} u^4 (-du) = \int_{-1/2}^{0} u^4 \, du = \left. \frac{u^5}{5} \right|_{-1/2}^{0} = \frac{1}{5}\left(0 - \left(-\frac{1}{2}\right)^5\right) = \frac{1}{5} \cdot \frac{1}{32} =$$

$$= \frac{1}{160}$$

12. (12pts) The equation $x + \sin x = 1$ is given.
a) Use the Intermediate Value Theorem to show that this equation has at least one real solution.
b) Use your calculator to find an interval of width 0.01 that contains your solution. Use IVT again to justify why the interval you found contains the solution.

$x + \sin x - 1 = 0$

Let $f(x) = x + \sin x - 1$

$f(0) = -1$

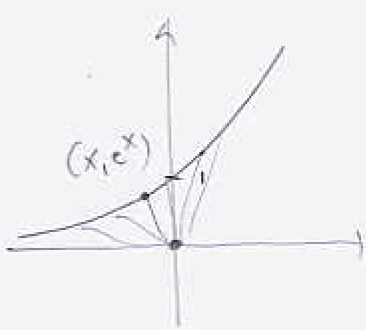
$f(2) = 1 + \sin 1 - 1 \approx 0.84$

Since $f(0) < 0 < f(1)$ by IVT there is a number c in $(0,1)$ so that $f(c) = 0$

x	$f(x)$
0.51	-0.0018
0.52	0.0168

Since $f(0.51) < 0 < f(0.52)$, by IVT there is a number c in $(0.51, 0.52)$ so that $f(c) = 0$.

Bonus. (14pts) Find the point on the graph of $y = e^x$ that is closest to the origin. Show that the point you find is, indeed, the closest. (Note: an equation will arise that you will not be able to solve algebraically, so use your calculator to get an approximate solution.)



$d\left((x, e^x), (0,0)\right) = \sqrt{(x-0)^2 + (e^x - 0)^2} = \sqrt{x^2 + e^{2x}}$

Enough to minimize $f(x) = d^2 = x^2 + e^{2x}$

Job: minimize f on $(-\infty, \infty)$

$f'(x) = 2x + 2e^{2x}$

$2x + 2e^{2x} = 0$

$x + e^{2x} = 0$

Using calculator, we get $x \approx -0.426303$

$f''(x) = 2 + 4e^{2x} > 0$ for all x

Thus, $x = -0.426303$ is a local min, since it is the only critical point, it must be an absolute min.

Point is $(-0.426303, 0.652919)$