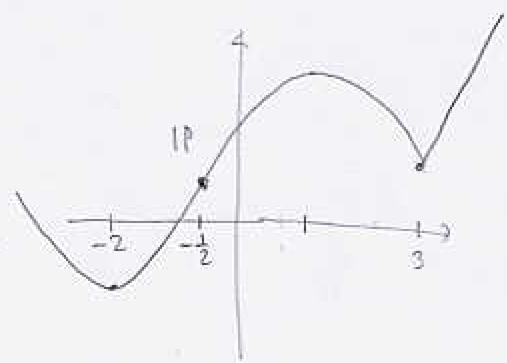
Spring '08/MAT 250/Exam 3 **Name:** Solution *Show all your work.*

1. (12pts) Draw the graph of a continuous function whose domain is **R** which satisfies the conditions below. (A sign chart may be helpful.)

$f'(x) > 0$ for $-2 < x < 1$, $f'(x) = 2$ for $x > 3$
$f'(x) < 0$ for $x < -2$, and $1 < x < 3$
$f'(3)$ does not exist
$f''(x) > 0$ for $x < -\frac{1}{2}$
$f''(x) < 0$ for $-\frac{1}{2} < x < 3$

	-2		1		3	
f'	$-$		$+$		$-$	2
f	$\searrow$	loc min	$\nearrow$	loc max	$\searrow$	$\nearrow$

		$-\frac{1}{2}$		3	
f''	$+$		$-$		0
f	CU	IP	CD		line



2. (10pts) Use L'Hospital's rule to find the limit:

$$\lim_{x\to 0} \frac{\sin(2x) - 2x}{x^3} \overset{LH}{=} \lim_{x\to 0} \frac{\cos(2x)\cdot 2 - 2}{3x^2} \overset{LH}{=} \lim_{x\to 0} \frac{-2\sin(2x)\cdot 2}{6x} \overset{LH}{=} \lim_{x\to 0} \frac{-8\cos(2x)}{6}$$

$\frac{\sin 0 - 0}{0} = \frac{0}{0}$ $\qquad \frac{1\cdot 2 - 2}{3\cdot 0} = \frac{0}{0}$ $\qquad \frac{0}{0}$

$$= -\frac{8}{6} = -\frac{4}{3}$$

3. (10pts) Use linearization (or differentials) to estimate $\sqrt{4.3}$. By how much does your estimate differ from the actual value?

$f(x) = \sqrt{x}$, $\quad f'(x) = \frac{1}{2\sqrt{x}}$

$f(4.3) \approx f(4) + f'(4)\,dx \approx f(4) + f'(4)\,dx$

$\approx \sqrt{4} + \frac{1}{2\sqrt{4}} \cdot 0.3 = 2 + \frac{1}{4} \cdot 0.3 = 2.075$

Actual: $\sqrt{4.3} = 2.073644$

Difference $\sqrt{4.3} - 2.075 = -0.0013558$

4. (14pts) Consider the function $f(x) = x^3 - 4x^2 + 2x + 7$ on the interval $[0, 1]$.
a) Verify the hypotheses of the Mean Value Theorem.
b) Verify the conclusion of the Mean Value Theorem.

a) f is a polynomial, thus, continuous and differentiable on $\mathbb{R}$, and hence on $[0,1]$

b) $\frac{f(1)-f(0)}{1-0} = \frac{(1-4+2+7)-7}{1} = \frac{-1}{1} = -1$

MVT says there is a c in $(0,1)$ so that $f'(c) = -1$

$f'(x) = 3x^2 - 8x + 2$

$3x^2 - 8x + 2 = -1$

$3x^2 - 8x + 3 = 0$

$x = \frac{8 \pm \sqrt{64 - 4 \cdot 3 \cdot 3}}{2 \cdot 3} = \frac{8 \pm \sqrt{28}}{6}$

$= \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3} = \begin{matrix} 2.21\ldots \\ 0.451\ldots \end{matrix}$

in $(0,1)$

5. (12pts) Find the absolute minimum and maximum values for the function $f(x) = x \ln x$ on the interval $[\frac{1}{5}, 5]$.

$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1$

$\ln x + 1 \overset{\cdot}{=} 0$

$\ln x = -1$

$x = e^{-1} = \frac{1}{e} = 0.36\ldots$

in $(\frac{1}{5}, 5)$

x	$x \ln x$	
$\frac{1}{e}$	$\frac{1}{e} \ln \frac{1}{e} = -\frac{1}{e} = -0.36$	abs. min
$\frac{1}{5}$	$\frac{1}{5} \ln \frac{1}{5} = -0.321888$	
5	$5 \ln 5 = 8.047190$	abs. max

6. (24pts) Let $f(x) = \dfrac{1}{1+e^x}$.

a) Find the horizontal asymptotes.

b) Find the intervals of increase/decrease.

c) Find the intervals where f is concave up or down and where it has an inflection point.

d) Use your calculator and the results of a), b) and c) to accurately sketch the graph of f.

a) $\lim\limits_{x\to\infty} \dfrac{1}{1+e^x} = \dfrac{1}{1+\infty} = 0$ $y=0$ horizontal asymptotes

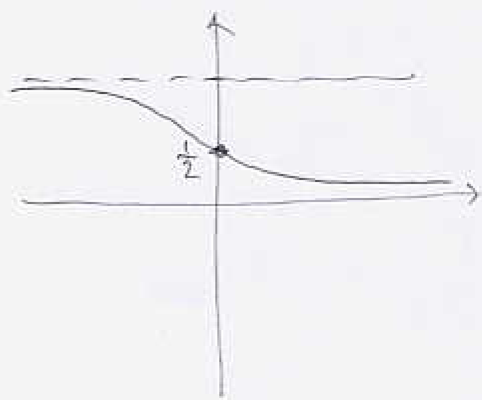
$\lim\limits_{x\to-\infty} \dfrac{1}{1+e^x} = \dfrac{1}{1+0} = 1$ $y=1$

$$f'(x) = \frac{d}{dx}(1+e^x)^{-1} = -(1+e^x)^{-2}\cdot e^x = -\frac{e^x}{(1+e^x)^2}$$

$$f''(x) = -\frac{e^x(1+e^x)^2 - e^x\,2(1+e^x)\cdot e^x}{(1+e^x)^4}$$

$$= -\frac{e^x(1+e^x)(1+e^x-2e^x)}{(1+e^x)^4}$$

$$= -\frac{e^x(1-e^x)}{(1+e^x)^3} = \frac{e^x(e^x-1)}{(1+e^x)^3}$$



b) $e^x > 0$ so $f'(x) < 0$

$(1+e^x)^2 > 0$ f' always decreasing

c) $e^x(e^x-1) = 0$ $1+e^x > 0$

so always defined

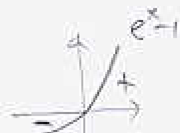
$e^x = 1$

$x = 0$

$e^x > 0$

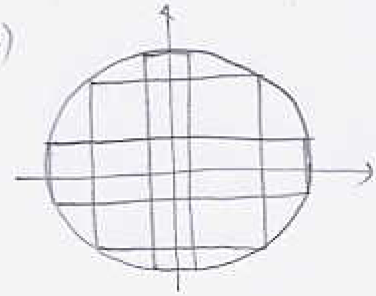
$(1+e^x)^3 > 0$

		0	
e^x-1	$-$	0	$+$
f	CD	IP	CU



7. (18pts) A rectangle is inscribed in the unit circle so that its sides are parallel to the axes.
a) Draw three rectangles, illustrating some of the ways this can be done.
b) Among all possible rectangles, find the dimensions of the one with the largest area.
Hints: let (x, y) be the vertex of the rectangle that is in the first quadrant. Express the area of the rectangle using x and y. What is the connection between x and y?

a)



$A = 2x \cdot 2y = 4xy = 4x\sqrt{1-x^2}$

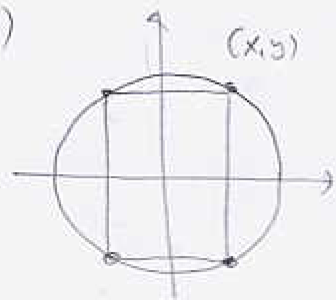
$x^2+y^2=1$ so $y=\pm\sqrt{1-x^2}$, can take $y=\sqrt{1-x^2}$

Job: maximize $f=x\sqrt{1-x^2}$ over $[0,1]$

$f'(x) = 1\sqrt{1-x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x)$

$= \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

b)



$0 \le x \le 1$

$1-2x^2=0$

$x^2 = \frac{1}{2}$

$x = \pm\sqrt{\frac{1}{2}}$

$x=\sqrt{\frac{1}{2}}$ in interval

x	$x\sqrt{1-x^2}$
$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}} = \frac{1}{2}$ abs. max
0	0
1	0

Max area occurs when $x = \sqrt{\frac{1}{2}}$

area is $4 \cdot \frac{1}{2} = 2$

Bonus. (10pts) Find the limit.

$$\lim_{x\to\infty}\left(1+\frac{4}{x}\right)^x = e^4$$

$y = \left(1+\frac{4}{x}\right)^x$

$\ln y = x\ln\left(1+\frac{4}{x}\right)$

$$\lim_{x\to\infty} x\ln\left(1+\frac{4}{x}\right) = \lim_{x\to\infty}\frac{\overset{\to 0}{\ln\left(1+\frac{4}{x}\right)}}{\underset{\to 0}{\frac{1}{x}}} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+\frac{4}{x}}\cdot\left(-\frac{4}{x^2}\right)}{-\frac{1}{x^2}}\cdot\frac{x^2}{x^2}$$

$$= \lim_{x\to\infty}\frac{4}{1+\frac{4}{x}} = \frac{4}{1+0} = 4$$