Spring '08/MAT 250/Exam 1 **Name:** Solution *Show all your work.*

1. (20pts) Find the following limits algebraically.

a) $\lim_{x\to-4} \frac{x^2-x-20}{x+4} = \lim_{x\to-4} \frac{\cancel{(x+4)}(x-5)}{\cancel{x+4}} = -4-5=-9$

b) $\lim_{x\to\infty} \frac{x^3+5x-1}{(x^2+1)(x^2-3)} = \lim_{x\to\infty} \frac{x^3\left(1+\frac{5}{x^2}-\frac{1}{x^3}\right)}{x^2\left(1+\frac{1}{x^2}\right)x\left(1-\frac{3}{x^2}\right)} = \lim_{x\to\infty} \underbrace{\frac{1}{x}}_{\to 0} \cdot \frac{\left(1+\overbrace{\frac{5}{x^2}}^{\to 0}-\overbrace{\frac{1}{x^3}}^{\to 0}\right)}{\left(1+\underbrace{\frac{1}{x^2}}_{\to 0}\right)\left(1-\underbrace{\frac{3}{x^2}}_{\to 0}\right)}$

$= 0 \cdot \frac{1+0-0}{(1+0)(1-0)} = 0$

c) $\lim_{x\to5} \frac{\sqrt{x}-\sqrt{5}}{x-5} = \lim_{x\to5} \frac{(\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})}{(x-5)(\sqrt{x}+\sqrt{5})} = \lim_{x\to5} \frac{\cancel{x-5}}{\cancel{(x-5)}(\sqrt{x}+\sqrt{5})}$

$= \lim_{x\to5} \frac{1}{\sqrt{x}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$

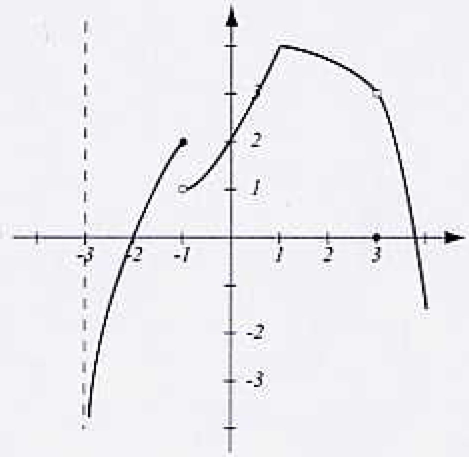
2. (16pts) Use the graph of the function to answer the following. Justify your answer if a limit does not exist.

$\lim_{x \to -3^+} f(x) = -\infty$

$\lim_{x \to -1^-} f(x) = 1$

$\lim_{x \to -1^+} f(x) = 2$

$\lim_{x \to 3} f(x) = 3$



List points where f is not continuous and justify why it is not continuous at those points.

f not continuous at $x=-1$, $x=3$

At $x=-1$: $\lim_{x \to -1} f(x)$ d.n.e.

At $x=3$: $\lim_{x \to 3} f(x) = 3$ but $f(3)=0$, not equal

At which point is f continuous but not differentiable? Why?

At $x=1$. There is a "sharp" point there.

3. (16pts) The equation $x^3 = 7$ is given.
a) Use the Intermediate Value Theorem to show that this equation has at least one real solution. (Incindentally, this proves the existence of $\sqrt[3]{7}$).
b) Use your calculator to find an interval of width 0.01 that contains your solution. Use IVT again to justify why the interval you found contains the solution.

a) $f(x) = x^3$

f is continuous as a polynomial

$f(0) = 0$

$f(2) = 8$

Since $f(0) < 7 < f(2)$ by IVT there is a c in $(0,2)$ so that $f(c) = 7$

b) $f(1.91) = 6.967$

$f(1.92) = 7.077$

Since $f(1.91) < 7 < f(1.92)$ by IVT there is a number c in $(1.91, 1.92)$ so that $f(c) = 7$.

4. (10pts) Find $\lim_{x\to 0} \sin x \cos\left(\frac{1}{x}+\frac{1}{x^2}\right)$. Use the theorem that rhymes with what polite people say when requesting something.

$$-1 \le \cos\left(\frac{1}{x}+\frac{1}{x^2}\right) \le 1 \quad | \sin x$$

$$-\sin x \le \sin x \cos\left(\frac{1}{x}+\frac{1}{x^2}\right) \le \sin x$$

$$\left.\begin{array}{l}\lim_{x\to 0} -\sin x = 0 \\ \lim_{x\to 0} \sin x = 0\end{array}\right\}$$ equal so by squeeze theorem

$$\lim_{x\to 0} \sin x \cos\left(\frac{1}{x}+\frac{1}{x^2}\right) = 0$$

5. (16pts) Let $f(x) = x^2 - 2x + 5$.
a) Find $f'(a)$.
b) Use a) to find the equation of the tangent line to the graph of f at the point $(2, 5)$.

a) $$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2-2x+5-(a^2-2a+5)}{x-a}$$

$$= \lim_{x\to a} \frac{x^2-a^2-2x+2a}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)-2(x-a)}{x-a}$$

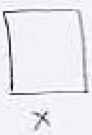
$$= \lim_{x\to a} \frac{\cancel{(x-a)}(x+a-2)}{\cancel{x-a}} = \lim_{x\to a} (x+a-2) = 2a-2$$

b) $$f'(2) = 2\cdot 2 - 2 = 2$$

$$y - 5 = 2(x-2)$$

$$y = 2x+1$$

6. (12pts) Let $f(x)$ denote the output in Watts of a square solar array whose side equals x meters.
a) What does $f'(x)$ represent and what are its units?
b) Suppose $f(3) = 120$ and $f'(3) = 11$. Use this information to estimate $f(3.2)$.

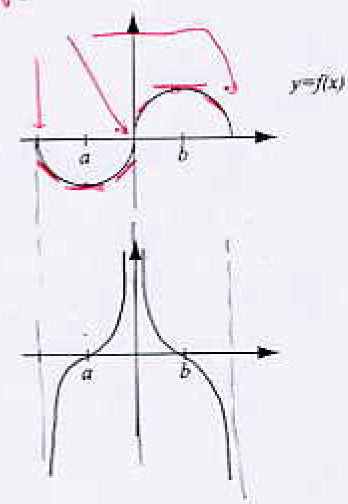


a) $f'(x)$ = rate of change of output w.r.t side length
units: Watts/meter

b) $f'(3) = 11$ means that for every additional meter of length, wattage increases by 11 watts. For an additional 0.2 meters, wattage increases $0.2 \cdot 11 = 2.2$ W. Thus $f(3.2) \approx 120 + 2.2 = 122.2$

7. (10pts) The graph of $f(x)$ is given. Draw the graph of $f'(x)$ under the graph of $f(x)$.

vertical tan. lines, f' not defined



Bonus. (10pts) Sketch the graph of a function that is continuous on $[-3, \infty)$ and satisfies the conditions given below.

x	-2	1	3
$f(x)$	2	-1	1

x	-2	1	3
$f'(x)$	-1	d.n.e.	0

$\lim_{x\to\infty} f(x) = 0$