Spring '07/MAT 335/Exam 1 **Name:** *Show all your work.*

1. (6pts) For the matrices A, B and C find the following expressions, if they are defined:
a) A^2C b) BB^T c) $2C - BA$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 0 & 1 \\ -2 & 3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

2. (6pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to effortlessly solve the system below.

$$A = \begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix}$$

$$\begin{array}{rrl} 2x_1 & +4x_2 & = 1 \\ 7x_1 & -x_2 & = 3 \end{array}$$

3. (4pts) Find the cosine of the angle between the vectors $\mathbf{a} = (1, -1, 3, 4)$ and $\mathbf{b} = (0, 4, 5, 2)$.

4. (9pts) A system of linear equations is given below.
a) Use the Gauss-Jordan method (that is, transform the augmented matrix to reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.
c) Describe the set of points in $\mathbf{R}^4$ that the solution set represents.

$$\begin{array}{rrrrl} 3x_1 & +x_2 & & +13x_4 & = 11 \\ & -x_2 & -x_3 & -6x_4 & = -1 \\ 2x_1 & +2x_2 & +x_3 & +17x_4 & = 9 \end{array}$$

5. (5pts) Below is the augmented matrix of a system of linear equations. Determine the c's for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \begin{bmatrix} 1 & 3 & 4 & 5+c \\ 0 & 1 & -17 & 7 \\ 0 & 0 & c^2-4c & c-4 \end{bmatrix}$$

6. (3pts) The matrix B was obtained by applying a row operation to matrix A. Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 3 & 7 & -7 \\ 1 & -6 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & -17 & 9 \\ 1 & -6 & 4 \end{bmatrix} \qquad E =$$

7. (3pts) Find a 2×2 matrix B so that $B \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 5a & 5b \end{bmatrix}$ for every 2×2 matrix.

8. (4pts) Suppose we have a system with 4 equations in 3 unknowns. Every equation represents a plane $\mathbf{R}^3$. Draw one example of a 4-plane arrangement for each of the following situations: a) the system has no solution b) the solution is a line in $\mathbf{R}^3$.

9. (10pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$
b) If A is a 3×5 matrix with at least 2 non-zero entries, then the solution set of the linear system $A\mathbf{x} = \mathbf{0}$ always has at most 3 parameters.
c) If A is an $n \times n$ matrix and $A^{17} = I$, then A is invertible.

Bonus. (5pts) Use a linear system to show that the vector $(-2, 25, 11)$ is a linear combination of vectors $(2, 1, 3)$ and $(3, -5, 1)$ and find the coefficients that realize this linear combination.

1. (5pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 3 & 2 & 3 & -1 & -4 \\ 4 & 7 & 2 & 3 & 15 \\ 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & -4 & -3 & 0 \\ 0 & 0 & 2 & -1 & 4 \end{vmatrix} =$$

2. (3pts) If $\det A = -5$ and A is a 2×2 matrix, find:

$\det A^{-1} =$

$\det(3A) =$

$\det A^4 =$

3. (6pts) Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose solution is given below (A is a 2×4 matrix).
a) Write any two solutions of the system.
b) Write the general solution of the system $A\mathbf{x} = \mathbf{0}$.
c) State the vectors that span the solution space of $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 &= 3 - 2s + 4t \\ x_2 &= 7 + 3s \\ x_3 &= -1 + 8s - 7t \\ x_4 &= -5s + t \end{aligned}$$

4. (6pts) Determine whether the vectors $(1, 3, 2)$, $(-2, 0, 7)$ and $(5, 3, -12)$ are linearly independent. Then draw a picture of these vectors that captures their relative positions to one another. Do not pay attention to actual coordinates.

5. (6pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$$

6. (4pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin by $120°$.
a) Write the standard matrix of this transformation.
b) Find $T(1, 3)$.

7. (7pts) Write the standard matrices for the following linear operators.
a) $T : \mathbf{R}^2 \to \mathbf{R}^2$, T dilates by 4 in the x-direction, then reflects in the line $y = x$.
b) $T : \mathbf{R}^3 \to \mathbf{R}^3$, T rotates about the positive z-axis by $90°$, then reflects in the xz-plane.

8. (4pts) Show that the set of vectors of form $(a, b, 3a - 2b)$ is a subspace of $\mathbf{R}^3$.

9. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\det A = 0$, then $\lambda = 1$ cannot be an eigenvalue of A.
b) If A is orthogonal, then $\det A \neq 0$.
c) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator, then $T(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

Bonus. (5pts) Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

Spring '07/MAT 335/Exam 3 **Name:** *Show all your work.*

1. (4pts) By inspection, explain why the following sets of vectors cannot be bases for $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively.
a) $\mathbf{v}_1 = (1,1)$, $\mathbf{v}_2 = (-1,2)$, $\mathbf{v}_3 = (0,1)$
b) $\mathbf{v}_1 = (1,2,0)$, $\mathbf{v}_2 = (0,2,1)$, $\mathbf{v}_3 = (1,0,-1)$

2. (5pts) Use matrix multiplication to find the matrix of the linear operator that is the composition of a rotation by 45° around the x axis, followed by a projection to the xy-plane.

3. (4pts) Find the standard matrix of the linear operator given by the equations below and determine whether it is a) one-to-one, or b) onto.

$$\begin{aligned} w_1 &= 5x_1 - 3x_2 \\ w_2 &= -x_1 + \tfrac{3}{5}x_2 \end{aligned}$$

4. (9pts) A matrix A is given below.
a) Find a basis for the row space of A.
b) Find a basis for the nullspace of A.
c) Verify that row(A)=null(A)$^\perp$ by showing that every basis vector for row(A) is orthogonal to every basis vector for null(A).

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

5. (5pts) Let W be the subspace of $\mathbf{R}^3$ spanned by vectors $(2,1,4)$ and $(1,-1,0)$. Find a basis for $W^\perp$.

6. (6pts) Let A be a 3×7 matrix. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) Give an example of a 3×7 matrix whose nullity is 5.

7. (4pts) Are the following vectors a basis for the subspace of $\mathbf{R}^5$ that they span?

$\mathbf{v}_1 = (*, *, *, *, 1)$, $\mathbf{v}_2 = (*, *, *, 1, 0)$, $\mathbf{v}_3 = (*, *, 1, 0, 0)$

8. (4pts) Complete the vector $(0, -1, 1)$ to a basis of $\mathbf{R}^3$. (That is, find additional vectors with which $(0, -1, 1)$ makes a basis.)

9. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If E is an elementary matrix, then A and EA have the same row space.
b) If A is a *nonzero* $m \times n$ matrix, then nullity$(A) \leq n - 1$.
c) For every 2×2 matrix A, row(A^T)=row$(A)^{\perp}$.

Bonus. (5pts) Let $\mathbf{v}_1 = (0, 3, -6, 5)$, $\mathbf{v}_2 = (0, 1, -2, 3)$. Write a linear system whose solution space is span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

1. (3pts) For the matrices A, B and C find the following expressions, if they are defined:
a) ABC b) BB^TC

$$A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

2. (8pts) A system of linear equations is given below.
a) Use the Gauss-Jordan method (that is, transform the augmented matrix to reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.
c) Write the solution of the homogeneous system (numbers on the right replaced by 0's). What is the basis of this subspace? What is the dimension?

$$\begin{array}{rrrrl} x_1 & -x_2 & +2x_3 & -x_4 & = -1 \\ 2x_1 & +x_2 & -2x_3 & -2x_4 & = -2 \\ -x_1 & +2x_2 & -4x_3 & +x_4 & = 1 \\ 3x_1 & & & -3x_4 & = -3 \end{array}$$

3. (4pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} =$$

4. (6pts) The matrix A is given below.
a) Find A^{-1}.
b) Use the result of a) to easily solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (-1, 2, 4)$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

5. (8pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.
c) Is there a basis of $\mathbf{R}^2$ consisting entirely of eigenvectors of A?

$$A = \begin{bmatrix} -5 & 7 \\ 1 & 3 \end{bmatrix}$$

6. (4pts) Are the vectors $(4, 1, 3)$, $(-2, 1, 8)$ and $(0, 1, -5)$ a basis for $\mathbf{R}^3$?

7. (4pts) Do the vectors of form (a, b, c) where $a + b + c = 1$ form a subspace of $\mathbf{R}^3$? Justify your answer.

8. (4pts) Find the matrix of the linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$ that is the composition of a rotation by 60° about the positive x-axis, followed by a projection to the xz-plane.

9. (5pts) Let T be the rotation about the origin in $\mathbf{R}^2$ by 30°. Find the vector that T sends to the vector $(-5, 3)$.

10. (8pts) Let A be a 5×3 matrix. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) If T_A is the linear transformation corresponding to A, is T_A ever onto? Is it ever one-to-one?

11. (3pts) Let E_1 be the matrix obtained from I_2 by adding 3 times row 1 to row 2 and let E_2 be be the matrix obtained from I_2 by swapping the two rows. Find the matrix below.

$$E_2E_1 \begin{bmatrix} 3 & -8 \\ 11 & -2 \end{bmatrix} =$$

12. (4pts) Let W be the subspace of $\mathbf{R}^4$ spanned by vectors $(1, 2, 1, 4)$ and $(3, 1, -1, 0)$. Find a basis for $W^\perp$.

13. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then $\mathbf{u} \cdot \mathbf{w} = 0$.
b) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator, then $T(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.
c) If A is an $n \times n$ matrix, then $\mathbf{0}$ is the only vector that is both in $\text{row}(A)$ and $\text{null}(A)$.

Bonus. (7pts) Let S be the set of vectors $S = \{(1, 2, -1), (4, 9, -6), (3, 7, -5), (6, 13, -8)\}$.
a) Find a basis for $\text{span}(S)$ that consists only of vectors in S.
b) Complete the basis you found in a) to a basis of $\mathbf{R}^3$.