Spring '07/MAT 335/Final Exam **Name:** Solution *Show all your work.*

1. (3pts) For the matrices A, B and C find the following expressions, if they are defined:
a) ABC b) BB^TC

$A = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$

$B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$

a) $\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & -7 \end{bmatrix}$

b) BB^TC is $\underbrace{(3\times2)(2\times3)(2\times2)}_{\text{not defined}}$

2. (8pts) A system of linear equations is given below.
a) Use the Gauss-Jordan method (that is, transform the augmented matrix to reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form.
c) Write the solution of the homogeneous system (numbers on the right replaced by 0's). What is the basis of this subspace? What is the dimension?

$$\begin{array}{rrrrl} x_1 & -x_2 & +2x_3 & -x_4 & = -1 \\ 2x_1 & +x_2 & -2x_3 & -2x_4 & = -2 \\ -x_1 & +2x_2 & -4x_3 & +x_4 & = 1 \\ 3x_1 & & & -3x_4 & = -3 \end{array}$$

÷3

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 1 & 0 & 0 & -1 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right] \to \begin{array}{ccccc} & & \text{free var} & & \\ & & \downarrow s & \downarrow t & \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right]$$

$x_1 - x_4 = -1$ $\quad x_1 = -1 + t$
$x_2 - 2x_3 = 0$ $\quad x_2 = 2s$
$x_3 = s$
$x_4 = t$

b) $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

c) $\vec{x} = s\begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

these vectors are the basis.

dimension = 2

3. (4pts) Evaluate the determinant by any (efficient) method:

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 2\left(2\cdot(4-1) - 1\cdot(2-0)\right) - 1\cdot 1\cdot(4-1)$$

$$= 2\cdot 4 - 3 = 5$$

4. (6pts) The matrix A is given below.

a) Find A^{-1}.

b) Use the result of a) to easily solve the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (-1, 2, 4)$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \leftarrow \div 2 \\ \\ \text{swap} \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & 3 & 0 & 1 & 0 \end{array}\right] \cdot(-4)$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -4 \end{array}\right] \begin{array}{l} + \\ \text{mult } b_3 \cdot -1 \end{array} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -1 & 4 \end{bmatrix}$$

b) $A\vec{x} = \vec{b}$

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -10 \\ 14 \end{bmatrix}$$

$\lambda^2+2\lambda-8=(\lambda-2)(\lambda+4)$

4-

5. (8pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.
c) Is there a basis of $\mathbf{R}^2$ consisting entirely of eigenvectors of A?

$$A = \begin{bmatrix} -5 & 7 \\ -1 & 3 \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda+5 & -7 \\ 1 & \lambda-3 \end{vmatrix} = (\lambda+5)(\lambda-3)-(-7) = \lambda^2+2\lambda-8$$

$\lambda^2+2\lambda-8=0 \qquad \lambda=-4, 2$

$(\lambda+4)(\lambda-2)=0$

b) $2I-A = \begin{bmatrix} 7 & -7 \\ 1 & -1 \end{bmatrix} \to [1 \ -1] \qquad x_2 = x_1 \qquad \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$-4I-A = \begin{bmatrix} 1 & -7 \\ 1 & -7 \end{bmatrix} \to [1 \ -7] \qquad x_1 = 7x_2 \qquad \vec{x} = t\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

c) Vectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^2$ (not proportional, there are two of them)

6. (4pts) Are the vectors $(4,1,3)$, $(-2,1,8)$ and $(0,1,-5)$ a basis for $\mathbf{R}^3$?

They are a basis if $\det|\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3| \neq 0$

$$\begin{vmatrix} 4 & -2 & 0 \\ 1 & 1 & 1 \\ 3 & 8 & -5 \end{vmatrix} = 4(-5-8)-(-2)(-5-3) = -52+2(-8)$$

$= -68$, yes, a basis

Expand by 1st row

7. (4pts) Do the vectors of form (a,b,c) where $a+b+c=1$ form a subspace of $\mathbf{R}^3$? Justify your answer.

Let $\vec{u}_1=(a_1,b_1,c_1)$, $\vec{u}_2=(a_2,b_2,c_2)$ be of mentioned form

Then $a_1+b_1+c_1=1 \qquad \vec{u}_1+\vec{u}_2 = (a_1+a_2, b_1+b_2, c_1+c_2)$

$a_2+b_2+c_2=1$

$(a_1+a_2)+(b_1+b_2)+(c_1+c_2) = 2$, not 1,

$a_1+a_2+b_1+b_2+c_1+c_2=2$

so $\vec{u}_1+\vec{u}_2$ is not in set

Not closed under addition.

8. (4pts) Find the matrix of the linear operator $T : \mathbf{R}^3 \to \mathbf{R}^3$ that is the composition of a rotation by $60°$ about the positive x-axis, followed by a projection to the xz-plane.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{projection to } xz\text{-plane}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 60° & -\sin 60° \\ 0 & \sin 60° & \cos 60° \end{bmatrix}}_{\text{rotation by } 60° \text{ about pos. } x\text{-axis}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

9. (5pts) Let T be the rotation about the origin in $\mathbf{R}^2$ by $30°$. Find the vector that T sends to the vector $(-5, 3)$.

$$[T] = \begin{bmatrix} \cos 30° & -\sin 30° \\ \sin 30° & \cos 30° \end{bmatrix}$$

$$[T]\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$\vec{x} = \underbrace{[T]^{-1}}_{\text{rotation by } -30°} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \cos(-30°) & -\sin(-30°) \\ \sin(-30°) & \cos(-30°) \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{3}+3}{2} \\ \frac{-5+3\sqrt{3}}{2} \end{bmatrix}$$

10. (8pts) Let A be a 5×3 matrix. Answer the following and justify your answers.
a) What is the biggest rank(A) could be?
b) What is the smallest nullity(A) could be?
c) If T_A is the linear transformation corresponding to A, is T_A ever onto? Is it ever one-to-one?

a) rank $A \leq$ smaller of $\overset{5}{m}, \overset{3}{n}$

rank $A \leq 3$

b) rank(A) + nullity(A) = 3

nullity(A) = 3 − rank(A)

nullity $A \geq 0$

c) T_A is onto if rank $A = 5$, which cannot happen

T_A is one-to-one if nullity(A) = 0, which happens if rank(A) = 3.

It can happen: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

11. (3pts) Let E_1 be the matrix obtained from I_2 by adding 3 times row 1 to row 2 and let E_2 be be the matrix obtained from I_2 by swapping the two rows. Find the matrix below.

$$E_2E_1 \begin{bmatrix} 3 & -8 \\ 11 & -2 \end{bmatrix} = E_2 \begin{bmatrix} 3 & -8 \\ 20 & -26 \end{bmatrix} = \begin{bmatrix} 20 & -26 \\ 3 & -8 \end{bmatrix}$$

add 3·row 1 to row 2

swap rows

12. (4pts) Let W be the subspace of $\mathbf{R}^4$ spanned by vectors $\underbrace{(1,2,1,4)}_{\vec{a}_1}$ and $\underbrace{(3,1,-1,0)}_{\vec{a}_2}$. Find a basis for $W^\perp$.

Need $\vec{x}$ so that $\vec{a}_1 \cdot \vec{x} = 0$

$\vec{a}_2 \cdot \vec{x} = 0$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\cdot(-3)} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -5 & -4 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & \frac{4}{5} & \frac{12}{5} \end{bmatrix} \xrightarrow{\cdot(-2)} \begin{array}{cccc} & & s & t \\ \end{array} \begin{bmatrix} 1 & 0 & -\frac{3}{5} & -\frac{4}{5} \\ 0 & 1 & \frac{4}{5} & \frac{12}{5} \end{bmatrix}$$

$x_1 = \frac{3}{5}s + -\frac{4}{5}t$

$x_2 = -\frac{4}{5}s - \frac{12}{5}t$

$$\vec{x} = s\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{4}{5} \\ -\frac{12}{5} \\ 0 \\ 1 \end{bmatrix}$$

basis for $W^\perp$

13. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R}^3$, if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{w} = 0$, then $\mathbf{u} \cdot \mathbf{w} = 0$.

b) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator, then $T(\mathbf{x} \cdot \mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

c) If A is an $n \times n$ matrix, then $\mathbf{0}$ is the only vector that is both in row(A) and null(A).

a) False. 

$\vec{u} = (1,0,0)$ $\quad \vec{u} \cdot \vec{v} = 0$
$\vec{v} = (0,0,1)$ $\quad \vec{v} \cdot \vec{w} = 0$ $\quad$ yet $\vec{u} \cdot \vec{w} = 1$
$\vec{w} = (1,-1,0)$

b) False. Let $T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad T(\vec{e}_1) \cdot T(\vec{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \neq \vec{e}_1 \cdot \vec{e}_2 = 0$

c) True: $\quad$ null(A) = row$(A)^\perp$ and we know that always

$W \cap W^\perp = \{\vec{0}\}$ so only $\vec{0}$ can be in both subspaces.

Bonus. (7pts) Let S be the set of vectors $S = \{(1, 2, -1), (4, 9, -6), (3, 7, -5), (6, 13, -8)\}$.

a) Find a basis for span(S) that consists only of vectors in S.

b) Complete the basis you found in a) to a basis of $\mathbf{R}^3$.

a) $\begin{bmatrix} 1 & 4 & 3 & 6 \\ 2 & 9 & 7 & 13 \\ -1 & -6 & -5 & -8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 3 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & 3 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(-2)

leading 1's here, so these two vectors in original matrix are a basis for the column space.

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ -6 \end{bmatrix}$

b) $\begin{vmatrix} 1 & 4 & 0 \\ 2 & 9 & 0 \\ -1 & -6 & 1 \end{vmatrix} = -1 \cdot (9-8) = -1$ $\quad$ so $\vec{e}_3$ completes them to a basis.