Spring '07/MAT 335/Exam 3 Name: Solution *Show all your work.*

1. (4pts) By inspection, explain why the following sets of vectors cannot be bases for $\mathbf{R}^2$ and $\mathbf{R}^3$, respectively.

a) $\mathbf{v}_1 = (1,1)$, $\mathbf{v}_2 = (-1,2)$, $\mathbf{v}_3 = (0,1)$

b) $\mathbf{v}_1 = (1,2,0)$, $\mathbf{v}_2 = (0,2,1)$, $\mathbf{v}_3 = (1,0,-1)$

a) a basis for $\mathbb{R}^2$ has only two vectors

b) $\vec{v}_3 = \vec{v}_1 - \vec{v}_2$ so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is not linearly independent

2. (5pts) Use matrix multiplication to find the matrix of the linear operator that is the composition of a rotation by 45° around the x axis, followed by a projection to the xy-plane.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{proj. to } xy\text{-plane}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}}_{\text{rotation by } 45^\circ \text{ around } x\text{-axis}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

3. (4pts) Find the standard matrix of the linear operator given by the equations below and determine whether it is a) one-to-one, or b) onto.

$$\begin{aligned} w_1 &= 5x_1 - 3x_2 \\ w_2 &= -x_1 + \tfrac{3}{5}x_2 \end{aligned}$$

Matrix of lin. transf is $A = \begin{bmatrix} 5 & -3 \\ -1 & \frac{3}{5} \end{bmatrix}$

$\det A = 5 \cdot \frac{3}{5} - 3 = 0$

Since $\det A = 0$, we have a) T_A is not one-to-one

b) T_A is not onto

4. (9pts) A matrix A is given below.
a) Find a basis for the row space of A.
b) Find a basis for the nullspace of A.
c) Verify that row(A)=null(A)$^\perp$ by showing that every basis vector for row(A) is orthogonal to every basis vector for null(A).

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 0 & 0 & 5 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \leftarrow \vec{a}_1 \\ \leftarrow \vec{a}_2 \end{array}$$

a) Basis for row space: $\vec{a}_1, \vec{a}_2$

b)
$x_1 = -3x_2 - 4x_4$
$x_3 = -2x_4$
$x_2 = s$
$x_4 = t$

$$\vec{x} = \begin{bmatrix} -3s-4t \\ s \\ -2t \\ t \end{bmatrix} = s\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}_1} + t\underbrace{\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\vec{b}_2}$$

Basis for null(A) is $\vec{b}_1, \vec{b}_2$

c) $\vec{a}_1 \cdot \vec{b}_1 = -3+3 = 0 \qquad \vec{a}_2 \cdot \vec{b}_1 = 0+0 = 0$
$\vec{a}_1 \cdot \vec{b}_2 = -4+4 = 0 \qquad \vec{a}_2 \cdot \vec{b}_2 = -2+2 = 0$

5. (5pts) Let W be the subspace of $\mathbf{R}^3$ spanned by vectors $(2,1,4)$ and $(1,-1,0)$. Find a basis for $W^\perp$.

Need solution to

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0} \qquad \cdot(-2)$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{4}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{4}{3} \end{bmatrix}$$

$x_1 = -\frac{4}{3}x_3$
$x_2 = -\frac{4}{3}x_3$
$x_3 = t$

$$\vec{x} = t\begin{bmatrix} -\frac{4}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

basis for $W^\perp$

6. (6pts) Let A be a 3×7 matrix. Answer the following and justify your answers.
a) What is the biggest ran(A) could be?
b) What is the smallest nullity(A) could be?
c) Give an example of a 3×7 matrix whose nullity is 5.

a) $\text{ran}(A) \leq \min\{3,7\} = 3$

b) $\text{nullity}(A) + \text{ran}(A) = 7$

$\text{ran}\,A = 7 - \text{nullity}(A) \leq 3$

$\text{nullity}(A) \geq 7-3 = 4$

c) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

rank = 2 so nullity = 5

7. (4pts) Are the following vectors a basis for the subspace of $\mathbf{R}^5$ that they span?

$\mathbf{v}_1 = (*, *, *, *, 1)$, $\mathbf{v}_2 = (*, *, *, 1, 0)$, $\mathbf{v}_3 = (*, *, 1, 0, 0)$

$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

5th row $c_1 = 0$

4th row $*c_1 + c_2 = 0 \Rightarrow c_2 = 0$

3rd row: $*c_1 + *c_2 + c_3 = 0 \Rightarrow c_3 = 0$

So they are linearly independent, thus a basis for $\mathbb{R}^5$

8. (4pts) Complete the vector $(0, -1, 1)$ to a basis of $\mathbf{R}^3$. (That is, find additional vectors with which $(0, -1, 1)$ makes a basis.)

$\begin{bmatrix} 0 & & \\ -1 & \vec{v}_2 & \vec{v}_3 \\ 1 & & \end{bmatrix}$

need vectors so $\det \neq 0$

$\begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -1 \cdot (-1-0) = 1$

$(\vec{v}_1, \vec{e}_1, \vec{e}_3)$ make a basis.

9. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.

a) If E is an elementary matrix, then A and EA have the same row space.

b) If A is a nonzero $m \times n$ matrix, then nullity$(A) \leq n - 1$.

c) For every 2x2 matrix A, row(A^T)=row$(A)^\perp$.

a) True. EA is a matrix obtained by a row operation from A, hence it has the same row space.

b) $\underbrace{\text{rank}(A)}_{\geq 0} + \text{nullity}(A) = n$ so $\overset{\geq 0}{\text{nullity}(A)} \leq n$

Since A is nonzero, rank$(A) \geq 1$ so nullity$(A) \leq n-1$

c) False: $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

row $(A)^\perp = \{(1,1)\}^\perp$, row A^T = span $\{e_1\}$

= span $(1,-1)$

OR: any A with det $A \neq 0$

row $(A) = \mathbb{R}^2$, (row $A)^\perp = \{0\}$

row $(A^T) = \mathbb{R}^2$

Bonus. (5pts) Let $\mathbf{v}_1 = (0, 3, -6, 5)$, $\mathbf{v}_2 = (0, 1, -2, 3)$. Write a linear system whose solution space is span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

Let $W =$ span $\{\vec{v}_1, \vec{v}_2\}$ If we find a basis for $W^\perp$, $W = (W^\perp)^\perp$ so a sol. of the equation $B\vec{x} = \vec{0}$, where rows of B are basis vectors of $W^\perp$

$$\cdot(-3) \begin{bmatrix} 0 & 3 & -6 & 5 \\ 0 & 1 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$x_1 = s$
$x_2 = 2x_3$
$x_3 = t$
$x_4 = 0$

$\vec{y} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Thus, $\vec{v}_1, \vec{v}_2$ are solutions of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{bmatrix} \vec{x} = 0$ or

$x_1 = 0$
$2x_2 + x_3 = 0$