Spring '07/MAT 335/Exam 2 **Name:** Solution *Show all your work.*

1. (5pts) Evaluate the determinant by any (efficient) method:

$$\left|\begin{array}{cc|ccc} 3 & 2 & 3 & -1 & -4 \\ 4 & 7 & 2 & 3 & 15 \\ \hline 0 & 0 & 2 & 5 & 2 \\ 0 & 0 & -4 & -3 & 0 \\ 0 & 0 & 2 & -1 & 4 \end{array}\right| = \begin{vmatrix} 3 & 2 \\ 4 & 7 \end{vmatrix} \cdot \begin{vmatrix} 2 & 5 & 2 \\ -4 & -3 & 0 \\ 2 & -1 & 4 \end{vmatrix} = (3\cdot7-4\cdot2)\begin{vmatrix} 2 & 5 & 2 \\ 0 & 7 & 4 \\ 0 & -6 & 2 \end{vmatrix}$$

may evaluate blocks

expand by first column

$$= 13\cdot 2\cdot \begin{vmatrix} 7 & 4 \\ -6 & 2 \end{vmatrix}$$

$$= 26\cdot(7\cdot2+6\cdot4)$$

$$= 26\cdot 38$$

$$= 988$$

2. (3pts) If $\det A = -5$ and A is a 2×2 matrix, find:

$\det A^{-1} = \frac{1}{\det A} = -\frac{1}{5}$

$\det(3A) = 3^2\cdot\det A = -45$

$\det A^4 = (\det A)^4 = (-5)^4 = 625$

3. (6pts) Let $A\mathbf{x} = \mathbf{b}$ be a linear system whose solution is given below (A is a 2×4 matrix).
a) Write any two solutions of the system.
b) Write the general solution of the system $A\mathbf{x} = \mathbf{0}$.
c) State the vectors that span the solution space of $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 &= 3 - 2s + 4t \\ x_2 &= 7 + 3s \\ x_3 &= -1 + 8s - 7t \\ x_4 &= -5s + t \end{aligned}$$

a) $s=0, t=0$

$$\vec{x} = \begin{bmatrix} 3 \\ 7 \\ -1 \\ 0 \end{bmatrix}$$

$s=1, t=-1$

$$\vec{x} = \begin{bmatrix} -3 \\ 10 \\ 14 \\ -6 \end{bmatrix}$$

b) $$\vec{x} = \begin{bmatrix} -2s+4t \\ 3s \\ 8s-7t \\ -5s+t \end{bmatrix} = s\begin{bmatrix} -2 \\ 3 \\ 8 \\ -5 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

c) these vectors span the solution space of $A\vec{x}=\vec{0}$

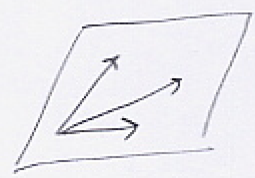
4. (6pts) Determine whether the vectors $(1,3,2)$, $(-2,0,7)$ and $(5,3,-12)$ are linearly independent. Then draw a picture of these vectors that captures their relative positions to one another. Do not pay attention to actual coordinates.

Check if $\begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 3 \\ 2 & 7 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has only the trivial solution.

$$\begin{bmatrix} 1 & -2 & 5 \\ 3 & 0 & 3 \\ 2 & 7 & -12 \end{bmatrix} \begin{matrix} -3 \\ \cdot(-2) \end{matrix} \to \begin{bmatrix} 1 & -2 & 5 \\ 0 & 6 & -12 \\ 0 & 11 & -22 \end{bmatrix} \underset{\text{swap}}{\to} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 6 & -12 \end{bmatrix} \cdot(-1 \to \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

free param

solution set has a free parameter so infinitely many solutions, therefore they are linearly dependent.



vectors all lie in a plane in $\mathbb{R}^3$

5. (6pts) The matrix A is given below.
a) Find the eigenvalues for the matrix.
b) For each eigenvalue, find a corresponding eigenvector.

$A = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix}$

a) $\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 \\ 1 & \lambda-5 \end{vmatrix} = (\lambda-3)(\lambda-5) - (-1)$

$= \lambda^2 - 8\lambda + 16 = (\lambda-4)^2$

$(\lambda-4)^2 = 0$, $\lambda = 4$ only eigenvalue

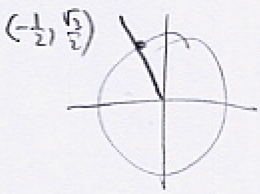
b) $4I - A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\quad x_1 = x_2 = t$, $\quad x_2 = t$

t

eigenvectors for $\lambda = 4$: $\begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

6. (4pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the rotation about the origin by $120°$.
a) Write the standard matrix of this transformation.
b) Find $T(1,3)$.

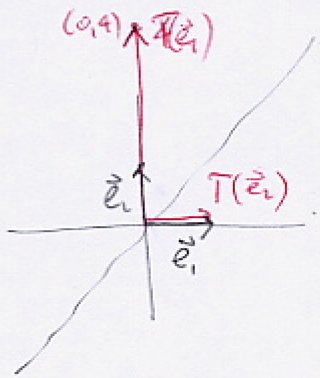
$$[T] = \begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$



$$T(1,3) = [T]\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{3\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - \frac{3}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1+3\sqrt{3}}{2} \\ \frac{\sqrt{3}-3}{2} \end{bmatrix}$$

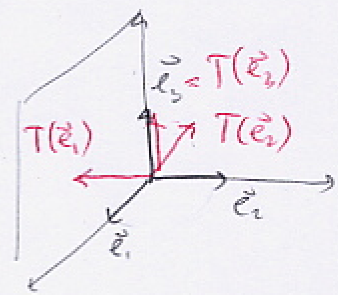
7. (7pts) Write the standard matrices for the following linear operators.
a) $T : \mathbf{R}^2 \to \mathbf{R}^2$, T dilates by 4 in the x-direction, then reflects in the line $y = x$.
b) $T : \mathbf{R}^3 \to \mathbf{R}^3$, T rotates about the positive z-axis by $90°$, then reflects in the xz-plane.

a)



$$[T] = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$$

b)



$$[T] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. (4pts) Show that the set of vectors of form $(a, b, 3a - 2b)$ is a subspace of $\mathbf{R}^3$.

Let $\vec{u} = (a_1, b_1, 3a_1 - 2b_1)$
$\vec{v} = (a_2, b_2, 3a_2 - 2b_2)$

$\vec{u} + \vec{v} = (a_1 + a_2, b_1 + b_2, 3a_1 - 2b_1 + 3a_2 - 2b_2)$
$= (a_1 + a_2, b_1 + b_2, 3(a_1 + a_2) - 2(b_1 + b_2)) \longrightarrow$ has form $(a, b, 3a - 2b)$

$c\vec{u} = (ca_1, cb_1, c(3a_1 - 2b_1)) = (ca_1, cb_1, 3ca_1 - 2cb_1)$

9. (9pts) Are the following statements true or false? Justify your answer by giving a logical argument or a counterexample.
a) If $\det A = 0$, then $\lambda = 1$ cannot be an eigenvalue of A.
b) If A is orthogonal, then $\det A \neq 0$.
c) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear operator, then $T(\mathbf{x}\cdot\mathbf{y}) = \mathbf{x}\cdot\mathbf{y}$ for every $\mathbf{x}$, $\mathbf{y}$ in $\mathbf{R}^2$.

a) False. $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvalues: 0, 1 (on diagonal) yet $\det A = 0$

b) (true) A orthogonal means $A^TA = I$ so A is invertible, which means $\det A \neq 0$

c) false: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $A\vec{e}_1 \cdot A\vec{e}_2 = 2$ yet $\vec{e}_1 \cdot \vec{e}_2 = 0$

$\uparrow$ $A\vec{e}_1$ $\uparrow$ $A\vec{e}_2$

Bonus. (5pts) Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y-x)(z-x)(z-y)$$

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} \cdot(-1) =$$

factor out $y-x$, $z-x$ from 2nd, 3rd row

$$= (y-x)(z-x)\underbrace{\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 0 & z-y \end{vmatrix}}_{\det = 1\cdot1\cdot(z-y)} = (y-x)(z-x)(z-y)$$