Spring '07/MAT 335/Exam 1 **Name:** Solution *Show all your work.*

1. (6pts) For the matrices A, B and C find the following expressions, if they are defined:
a) A^2C b) BB^T c) $2C - BA$

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 0 & 1 \\ -2 & 3 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

a) A^2 is not defined $(3\times 2)(3\times 2)$

b) $$\begin{bmatrix} 7 & 0 & 1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 7 & -2 \\ 0 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 50 & -12 \\ -12 & 17 \end{bmatrix}$$

c) $$\begin{bmatrix} 4 & 2 \\ -2 & -2 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 1 \\ -2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -2 & -2 \end{bmatrix} - \begin{bmatrix} 14 & 10 \\ -7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -8 \\ 5 & -3 \end{bmatrix}$$

2. (6pts) The matrix A is given below.
a) Find the inverse of A.
b) Use the inverse to effortlessly solve the system below.

$$A = \begin{bmatrix} 2 & 4 \\ 7 & -1 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &+ 4x_2 &= 1 \\ 7x_1 &- x_2 &= 3 \end{aligned}$$

a) $$A^{-1} = \frac{1}{2\cdot(-1) - 7\cdot 4}\begin{bmatrix} -1 & -4 \\ -7 & 2 \end{bmatrix} = -\frac{1}{30}\begin{bmatrix} -1 & -4 \\ -7 & 2 \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} 1 & 4 \\ 7 & -2 \end{bmatrix}$$

b) solution is $$A^{-1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{30}\begin{bmatrix} 1 & 4 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{30}\begin{bmatrix} 13 \\ 1 \end{bmatrix} = \begin{bmatrix} 13/30 \\ 1/30 \end{bmatrix}$$

3. (4pts) Find the cosine of the angle between the vectors $\mathbf{a} = (1, -1, 3, 4)$ and $\mathbf{b} = (0, 4, 5, 2)$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{0-4+15+8}{\sqrt{1+1+9+16}\cdot\sqrt{0+16+25+4}}$$

$$= \frac{19}{\sqrt{27}\,\sqrt{45}}$$

4. (9pts) A system of linear equations is given below.
a) Use the Gauss-Jordan method (that is, transform the augmented matrix to reduced row-echelon form) in order to solve the system.
b) Write the solution in vector form as well.
c) Describe the set of points in $\mathbf{R}^4$ that the solution set represents.

$$\begin{array}{rrrrl} 3x_1 & +x_2 & & +13x_4 & = 11 \\ & -x_2 & -x_3 & -6x_4 & = -1 \\ 2x_1 & +2x_2 & +x_3 & +17x_4 & = 9 \end{array}$$

$$\begin{bmatrix} 3 & 1 & 0 & 13 & 11 \\ 0 & -1 & -1 & -6 & -1 \\ 2 & 2 & 1 & 17 & 9 \end{bmatrix} \begin{array}{l} \\ (-1) \\ (-1) \end{array} \to \begin{bmatrix} 1 & -1 & -1 & -4 & 2 \\ 0 & 1 & 1 & 6 & 1 \\ 2 & 2 & 1 & 17 & 9 \end{bmatrix} (-2)$$

$$\begin{array}{l} \\ \cdot 1 \\ \cdot(-4) \end{array} \begin{bmatrix} 1 & -1 & -1 & -4 & 2 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 4 & 3 & 25 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 6 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix} \cdot 1 \quad \text{3rd row} \cdot (-1)$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 7 & 2 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right]$$

$x_4 = t$ $\quad x_1 = 3 - 2t$
$x_2 = 2 - 7t$
$x_3 = -1 + t$
$x_4 = t$

b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -7 \\ 1 \\ 1 \end{bmatrix}$$

c) Solution is a line in $\mathbb{R}^4$.

5. (5pts) Below is the augmented matrix of a system of linear equations. Determine the c's for which the system has: a) one solution, b) infinitely many solutions, c) no solutions. (Note: no row operations are needed.)

$$A = \begin{bmatrix} 1 & 3 & 4 & 5+c \\ 0 & 1 & -17 & 7 \\ 0 & 0 & c^2-4c & c-4 \end{bmatrix}$$

Suppose $c^2 - 4c = 0$

$c(c-4) = 0$

$c = 0$ or $c = 4$

If $c=4$, last row is zeroes, may be dropped, get infinitely many solutions

$c=0$, last row is $[0\ 0\ 0\ {-4}]$, inconsistent system

If $c \neq 0, 4$, may divide by c^2-4c to get last row $[0\ 0\ 1\ \frac{1}{c}]$

— one solution

6. (3pts) The matrix B was obtained by applying a row operation to matrix A. Find the elementary matrix E so that $EA = B$.

$$A = \begin{bmatrix} 3 & 7 & -7 \\ 1 & -6 & 4 \end{bmatrix} \cdot 4 \quad B = \begin{bmatrix} 7 & -17 & 9 \\ 1 & -6 & 4 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

add second row $\cdot 4$ to first row

7. (3pts) Find a 2×2 matrix B so that $B\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 5a & 5b \end{bmatrix}$ for every 2×2 matrix.

$$B = \begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix}$$

If not obvious, try

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3a & 3b \\ 5a & 5b \end{bmatrix}$$

$xa + yc = 3a$

$xb + yd = 3b$

$za + wc = 5a$

$zb + wd = 5b$

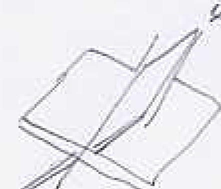
may guess:

$x = 3,\ y = 0$

$z = 5,\ w = 0$

8. (4pts) Suppose we have a system with 4 equations in 3 unknowns. Every equation represents a plane $\mathbf{R}^3$. Draw one example of a 4-plane arrangement for each of the following situations: a) the system has no solution b) the solution is a line in $\mathbf{R}^3$.

2 identical planes



2 identical planes

9. (10pts) Are the following statements always true or sometimes false? Justify your answer by giving a logical argument or a counterexample.
a) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$, then $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2$
b) If A is a 3×5 matrix with at least 2 non-zero entries, then the solution set of the linear system $A\mathbf{x} = \mathbf{0}$ always has at most 3 parameters.
c) If A is an $n \times n$ matrix and $A^{17} = I$, then A is invertible.

a) true

$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + \underbrace{2\vec{u}\cdot\vec{v}}_{=0} + \|\vec{v}\|^2$

$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \|\vec{u}\|^2 - \underbrace{2\vec{u}\cdot\vec{v}}_{=0} + \|\vec{v}\|^2$

are equal

b) False $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ has four parameters

$x_1 + x_2 + x_3 + x_4 + x_5 = 0$

c) True. $A \cdot A^{16} = I = A^{16} \cdot A$

so A^{16} is its inverse

Bonus. (5pts) Use a linear system to show that the vector $(-2, 25, 11)$ is a linear combination of vectors $(2, 1, 3)$ and $(3, -5, 1)$ and find the coefficients that realize this linear combination.

$(-2, 25, 11) = x_1(2,1,3) + x_2(3,-5,1)$

$= (2x_1 + 3x_2,\ x_1 - 5x_2,\ 3x_1 + x_2)$

$2x_1 + 3x_2 = -2$
$x_1 - 5x_2 = 25$
$3x_1 + x_2 = 11$

$\left[\begin{array}{cc|c} 2 & 3 & -2 \\ 1 & -5 & 25 \\ 3 & 1 & 11 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -5 & 25 \\ 2 & 3 & -2 \\ 3 & 1 & 11 \end{bmatrix} \begin{matrix} \cdot(-2) \\ \\ \cdot(-3) \end{matrix} \rightarrow \begin{bmatrix} 1 & -5 & 25 \\ 0 & 13 & -52 \\ 0 & 16 & -64 \end{bmatrix} \begin{matrix} \\ /13 \\ /16 \end{matrix}$

$\rightarrow \begin{bmatrix} 1 & -5 & 25 \\ 0 & 1 & -4 \\ 0 & 1 & -4 \end{bmatrix} \begin{matrix} \\ 5\cdot(1) \\ \text{drop} \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$

$x_1 = 5$
$x_2 = -4$