Spring '07/MAT 250/Final Exam **Name:** Solution *Show all your work.*

1. (6pts) Use the graph of the function to answer the following. Justify if a limit does not exist.

$\lim_{x \to -2} f(x) = 1$

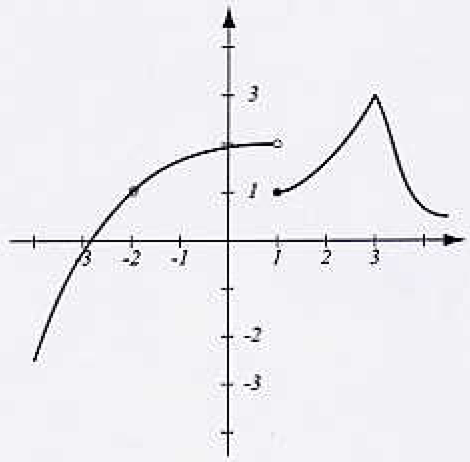
$\lim_{x \to 1} f(x) =$ d.n.e, one-sided limits are not equal

At which points is f discontinuous?

at $x=1$

At which points is f not differentiable?

at $x=1$ and $x=3$



2. (6pts) Find the limit $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 + x - 12}$ in two ways:

a) algebraically

b) Using L'Hospital's rule

a) $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 + x - 12} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+4)} = \frac{5}{7}$

b) $\lim_{x \to 3} \frac{\overset{\to 0}{x^2 - x - 6}}{\underset{\to 0}{x^2 + x - 12}} = \lim_{x \to 3} \frac{2x-1}{2x+1} = \frac{5}{7}$

3. (5pts) Use implicit differentiation to find y'.

$x^2 + y^2 = x^2y^2 \quad \Big| \frac{d}{dx}$

$$2x + 2yy' = 2xy^2 + x^2 \cdot 2yy'$$

$$2yy' - 2x^2yy' = 2xy^2 - 2x$$

$$y' \, 2(y - x^2y) = 2(xy^2 - x)$$

$$y' = \frac{2(xy^2 - x)}{2(y - x^2y)} = \frac{xy^2 - x}{y - x^2y}$$

4. (7pts) Let $P(t) = 150e^{0.04t}$ be the population (in thousands) of a certain city at time t, measured in years.

a) What is the population $t = 3$?

b) At what rate is the population increasing when $t = 3$? What are the units?

c) Interpret the meaning of the number in b) by approximating the population at time $t = 3.25$.

d) What is the exact population at time $t = 3.25$?

a) $P(3) = 150e^{0.04 \cdot 3} = 150e^{0.12} = 169.125$ thousand $\approx 169,125$ people

b) $P'(t) = 150e^{0.04t} \cdot 0.04 = 6e^{0.04t}$

$P'(3) = 6e^{0.12} = 6.765$ thousand/year

$\approx 6,765$ people/year

c) In 0.25 years, population should grow approx $0.25 \cdot 6.765 = 1,691$ people

so $P(3.25) \approx 169,125 + 1691 = 170,816$

d) $P(3.25) = 150e^{0.04 \cdot 3.25} \approx 170,824$

close

5. (9pts) Let $f(x) = \dfrac{1}{x^2+4}$.

a) Find the intervals of increase/decrease and where f has a local maximum and minimum.

b) Find the intervals where f is concave up or down.

c) Use your calculator and the results of a) and b) to accurately sketch the graph of f.

$$f'(x) = \frac{d}{dx}(x^2+4)^{-1} = -(x^2+4)^{-2}\cdot 2x = -\frac{2x}{(x^2+4)^2}$$

$$f''(x) = -\frac{2(x^2+4)^2 - 2x\cdot 2(x^2+4)\cdot 2x}{(x^2+4)^4} = -\frac{2(x^2+4)\overbrace{(x^2+4-4x^2)}^{4-3x^2}}{(x^2+4)^4} = \frac{2(3x^2-4)}{(x^2+4)^3}$$

a) crit pts: $f'=0$: $2x=0$, $x=0$; f' d.n.e.: $x^2+4=0$, $x^2=-4$, no sol

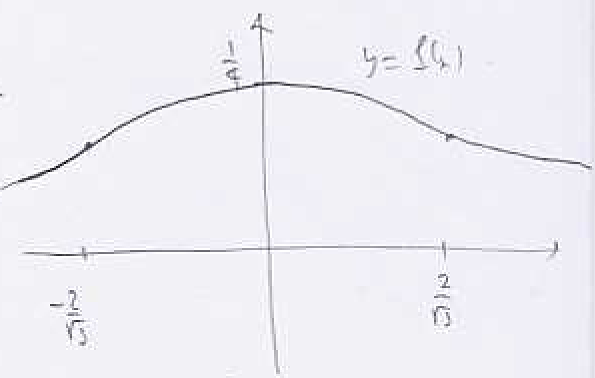
$$f'(x) = -\frac{2x}{(x^2+4)^2} \begin{matrix} \leftarrow \text{determines sign} \\ \leftarrow >0 \end{matrix}$$

		0	
$f'(x)$	+	0	−
$f(x)$	↗	loc. max	↘

b) $f''=0$: $3x^2-4=0$, $x = \pm\frac{2}{\sqrt3}$; f'' d.n.e.: no sol.

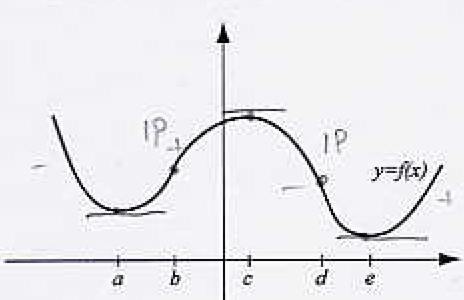
$$f''(x) = \frac{2(3x^2-4)}{(x^2+4)^3} \begin{matrix} \leftarrow \text{determines sign} \\ \leftarrow >0 \end{matrix}$$

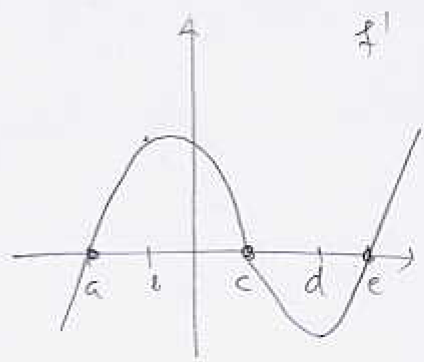
		$-\frac{2}{\sqrt3}$		$\frac{2}{\sqrt3}$	
f''	+	0	−	0	+
f	CU	infl.	CD	infl.	CU

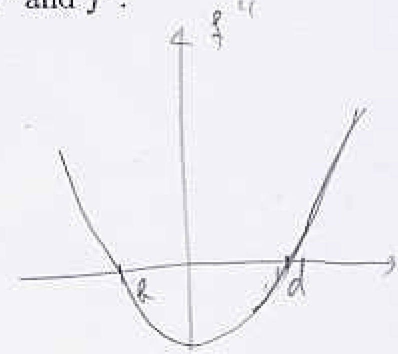


$$\frac{1}{\frac43+4} = \frac{3}{16}$$

6. (5pts) The graph of f is shown in the picture. Draw the graphs of f' and f''.







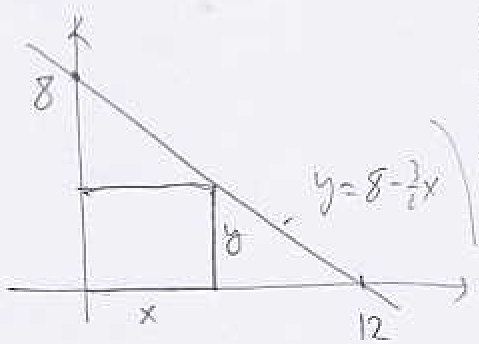
7. (5pts) Let $f(x) = \sqrt{x}$. Find $f'(x)$ using the definition of the derivative. Then compare to what you get using a differentiation rule.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} =$$

$$= \lim_{h\to 0} \frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

8. (7pts) The line $y = 8 - \frac{2}{3}x$ makes a triangle with the x- and y-axes. A rectangle may be inscribed in this triangle so that one vertex is on the line and two sides lie on the axes. Among all such rectangles, find the one that has the greatest area.



$8 - \frac{2}{3}x = 0$

$\frac{2}{3}x = 8$

$x = 12$

$0 \le x \le 12$

$A = xy = x\left(8 - \frac{2}{3}x\right) = -\frac{2}{3}x^2 + 8x$

Job: maximize $A(x) = -\frac{2}{3}x^2 + 8x$ on $[0, 12]$

$A'(x) = -\frac{4}{3}x + 8$

$-\frac{4}{3}x + 8 = 0 \qquad x = 6$

$\frac{4}{3}x = 8$

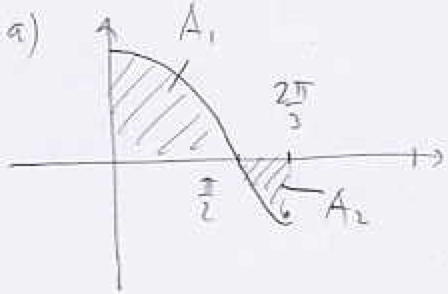
x	$A(x)$
0	0
12	0
6	24

$-\frac{2}{3} \cdot 36 + 48$

$= 24$

so at $x = 6$ there is an absolute max.

9. (6pts) Consider the integral $\int_0^{\frac{2\pi}{3}} \cos x\, dx$.
a) Use a picture to determine whether this definite integral is positive or negative.
b) Evaluate the integral and verify your conclusion from a).

a) 

$\int_0^{\frac{2\pi}{3}} \cos x\, dx = A_1 - A_2 > 0$ since, from picture, $A_1 > A_2$

b) $\int_0^{\frac{2\pi}{3}} \cos x\, dx = \sin x \Big|_0^{\frac{2\pi}{3}} = \sin \frac{2\pi}{3} - \sin 0 = \frac{\sqrt{3}}{2} > 0$

10. (5pts) Use substitution (don't forget to change bounds) to evaluate:

$$\int_0^1 x^2(1+2x^3)^5\, dx = \left[\begin{array}{ll} u = 1+2x^3 & x=1,\ u=3 \\ du = 6x^2\, dx & x=0,\ u=1 \\ \frac{1}{6}du = x^2\, dx & \end{array}\right]$$

$$= \int_1^3 u^5 \frac{1}{6}\, du = \frac{1}{6}\cdot\frac{u^6}{6}\Big|_1^3 = \frac{1}{36}(3^6-1) = \frac{728}{36} = \frac{182}{9}$$

11. (5pts) Simplify using part 1 of the Fundamental Theorem of Calculus. (Hint: if $g(y) = \int_2^{y} t^2 \sin t\, dt$, you are trying to find the derivative of $g(y^2)$. What is $g'(y)$?)

$$\frac{d}{dy}\int_2^{y^2} t^2 \sin t\, dt = \frac{d}{dy}\left(g(y^2)\right) = g'(y^2)\cdot 2y = (y^2)^2 \sin y^2 \cdot 2y$$

$$= 2y^5 \sin y^2$$

$g(y) = \int_2^y t^2 \sin t\, dt$

$g'(y) = y^2 \sin y$

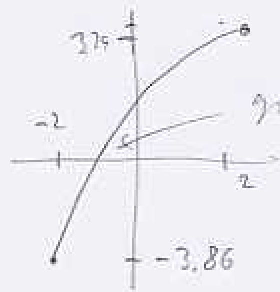
12. (4pts) Use the Intermediate Value Theorem to show that the equation $e^x = x^2$ has a solution in $[-2, 2]$.

$e^x - x^2 = 0$

$f(-2) = e^{-2} - 4 = -3.86 < 0$

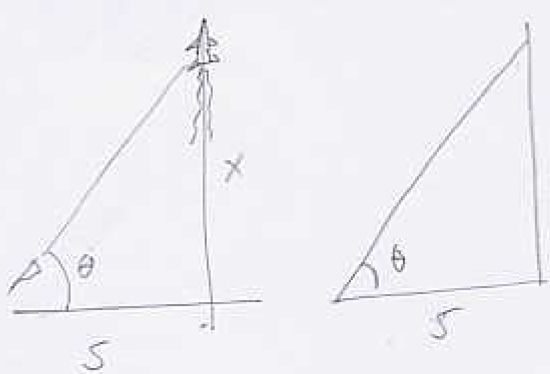
$f(2) = e^2 - 4 = 3.39 > 0$

Since $-3.86 < 0 < 3.39$ by IVT there exists a number c in $[-2, 2]$ so that $f(c) = 0$



graph must cross x-axis somewhere

Bonus. (7pts) A camera 5km away from the launch pad tracks a rocket as it lifts off. As the rocket moves upwards, the camera keeps it in its view. When the angle with the horizontal of the camera is $\frac{\pi}{4}$ radians, the angle is increasing at rate 0.1 radians per second. What is the speed of the rocket at that moment?



Know: $\frac{d\theta}{dt} = 0.1$ rad/sec

Need: $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$

$\frac{x}{5} = \tan\theta \quad \Big| \frac{d}{dt}$

$\frac{x'}{5} = \sec^2\theta \cdot \theta'$

$x' = 5\sec^2\theta \cdot \theta'$

$x' = 5\sec^2\frac{\pi}{4} \cdot 0.1 = 5 \cdot 2 \cdot 0.1 = 1$ km/s

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$