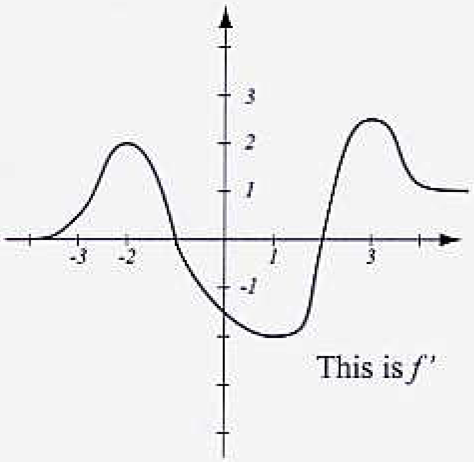
Spring '07/MAT 250/Exam 3 **Name:** Solution *Show all your work.*

1. (7pts) The graph of the function f' is given. Answer the following questions about f, which is defined for all real numbers (note: questions are not about f'). You may use a sign chart if it is helpful.

a) On which intervals is f increasing/decreasing?

b) On which intervals if f concave up/concave down?

c) At which points does f have local maxima/minima?



a), c)

$$\begin{array}{cccccc} & & -1 & & 2 & \\ \hline f' & + & 0 & - & 0 & + \\ f & \nearrow & \text{loc max} & \searrow & \text{loc. min} & \nearrow \end{array}$$

loc. max at $x=-1$

loc min at $x=2$

b) f conc up if f' incr, on $(-\infty,-2)$ and $(1,3)$

f conc. down if f' decr, on $(-2,1)$ and $(3,\infty)$

2. (10pts) Use L'Hospital's rule to find the limits:

a) $$\lim_{x\to0}\frac{\overset{\to\, 1-1-0=0}{e^x-1-x}}{\underset{\to 0}{x^2}} \underset{L'H}{=} \lim_{x\to0}\frac{\overset{\to 0}{e^x-1}}{\underset{\to 0}{2x}} = \lim_{x\to0}\frac{e^x}{2} = \frac{1}{2}$$

b) $$\lim_{x\to0^+} x^5\ln x = \lim_{x\to0+}\frac{\overset{\to -\infty}{\ln x}}{\underset{\to\infty}{\frac{1}{x^5}}} = \lim_{x\to0+}\frac{\frac{1}{x}}{-5x^{-6}} = \lim_{x\to0+} -\frac{\frac{1}{x}\cdot x^6}{5} =$$

$$= \lim_{x\to0+} -\frac{x^5}{5} = \frac{0}{5} = 0$$

5. (11pts) Let $f(x) = x^3 e^x$.

a) Find the intervals of increase/decrease and where f has a local maximum and minimum.

b) Find the intervals where f is concave up or down.

c) Use your calculator and the results of a) and b) to accurately sketch the graph of f.

a) $f'(x) = 3x^2e^x + x^3e^x$

$= e^x(x^3+3x^2)$

$f''(x) = e^x(x^3+3x^2) + e^x(3x^2+6x)$

$= e^x(x^3+6x^2+6x)$

$= e^x \cdot x(x^2+6x+6)$

critical pts:

$\underbrace{e^x}_{>0}(x^3+3x^2) = 0$

$x^3+3x^2=0$

$x^2(x+3)=0$

$x = 0, -3$

		−3		0	
f'	−	0	+	0	+
f	↘		↗		↗

critical for f'':

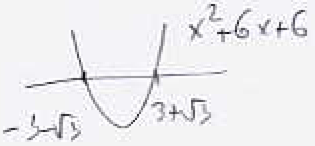
$\underbrace{e^x}_{>0} \cdot x(x^2+6x+6) = 0$

$x = 0$ or $x^2+6x+6=0$

$x = \frac{-6 \pm \sqrt{36-4\cdot1\cdot6}}{2}$

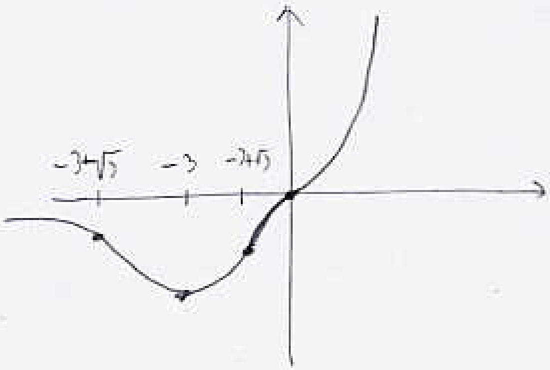
$= \frac{-6 \pm \sqrt{12}}{2} = \frac{-6 \pm 2\sqrt{3}}{2}$

$= -3 \pm \sqrt{3} = -4.73, -1.27$



$f'(x) = e^x x^2(x+3)$ has sign of $x+3$

		$-3-\sqrt{3}$		$-3+\sqrt{3}$		0	
x	−		−		−		+
x^2+6x+6	+	0	−	0	+		+
$f''(x)$	−	0	+	0	−	0	+
f	CD	IP	CU	IP	CD	IP	CU



3. (7pts) Consider the function $f(x) = x^2 - 7x - 3$ on the interval $[2, 6]$.
a) Verify the hypotheses of the Mean Value Theorem.
b) Verify the conclusion of the Mean Value Theorem.

a) $f(x)$ is a polynomial, continuous on $\mathbb{R}$

$f'(x) = 2x-7$ defined for any x.

b) $\frac{f(6)-f(2)}{6-2} = \frac{(36-42-3)-(4-14-3)}{4} = \frac{-9-(-13)}{4} = \frac{4}{4} = 1$

MVT says there exists a c in $(2,6)$ so that $f'(x)=1$

$2x-7=1$

$2x=8$

$x=4$ ← is in $(2,6)$

4. (7pts) Find the absolute minimum and maximum values for the function $f(x) = 4x - \tan x$ on the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

$f'(x) = 4 - \sec^2 x$

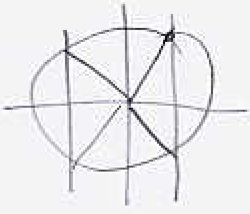
$4 - \sec^2 x = 0$

$\sec^2 x = 4$

$\frac{1}{\cos x} = \sec x = \pm 2$

$\cos x = \pm \frac{1}{2}$ $\quad x = \frac{\pi}{3}, -\frac{\pi}{3}$

$\frac{2\pi}{3}, -\frac{2\pi}{3}$



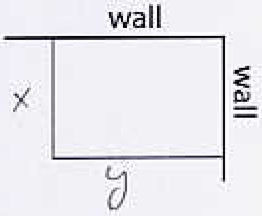
none in $[-\frac{\pi}{4}, \frac{\pi}{4}]$

(-2.14)

x	$f(x)$	
$-\frac{\pi}{4}$	$-4\frac{\pi}{4} - \tan(-\frac{\pi}{4}) = -\pi + 1$	abs min
$\frac{\pi}{4}$	$4\frac{\pi}{4} - \tan\frac{\pi}{4} = \pi - 1$	abs max

(2.14)

6. (8pts) Sheila wishes to enclose a rectangular play pen so its area is 12m^2. Two sides of the pen are walls (see picture) and a fence is used for the remaining two sides. Find the dimensions of the pen that minimize the length of the fence. Show that the number you find, does, indeed, give you a minimum length.



$xy=12 \quad y=\frac{12}{x}$

length $= x+y = x+\frac{12}{x}$

Job: minimize $\ell(x) = x+\frac{12}{x}$ on $(0,\infty)$

$\ell'(x) = 1-\frac{12}{x^2}$ $\quad$ $\ell''(x) = -\frac{12\cdot(-2)}{x^3} = \frac{24}{x^3}$

$1-\frac{12}{x^2}=0$ $\quad$ $\ell''(\sqrt{12}) = \frac{24}{(\sqrt{12})^3} > 0$ ∪ conc up

$x^2=12$

$x = \pm\sqrt{12}$

not in interval

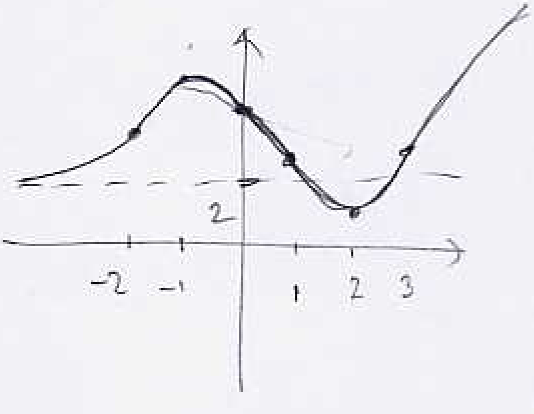
so have a local min at $x=\sqrt{12}$

Because it is the only critical pt, it must be an abs. min.

Bonus. (5pts) Use information you gathered in problem 1 to draw the graph of the function f if it is known that f satisfies the additional conditions:

$f(0)=4$

$\lim_{x\to-\infty} f(x) = 2$



What is $\lim_{x\to\infty} f(x)$?

$\lim_{x\to\infty} f(x) = \infty$ since when $x \geq 3$, $f'(x) \geq 1$

so $f'(x)$ grows at least as fast as x.