Spring '07/MAT 250/Exam 2 **Name:** Solution *Show all your work.*

Differentiate and simplify where appropriate:

1. (4pts) $\frac{d}{dx}(7x^7 - \frac{1}{\sqrt[4]{x^3}} - \frac{7}{x^4} + e) = 49x^6 + \frac{3}{4}x^{-\frac{7}{4}} + 28x^{-5}$

$x^{-\frac{3}{4}} - 7x^{-4}$ constant

2. (4pts) $\frac{d}{dx}x^{10}e^{3x} = 10x^9 e^{3x} + x^{10}e^{3x}\cdot 3$

$= e^{3x}(10x^9 + 3x^{10})$

3. (4pts) $\frac{d}{dx}\frac{x^2+4}{3x-7} = \frac{2x(3x-7)-(x^2+4)\cdot 3}{(3x-7)^2} = \frac{6x^2-14x-3x^2-12}{(3x-7)^2}$

$= \frac{3x^2-14x-12}{(3x-7)^2}$

4. (5pts) $\frac{d}{dx}\ln\left(\frac{2x+1}{3x-7}\right)^4 = \frac{d}{dx} 4\ln\left(\frac{2x+1}{3x-7}\right) = \frac{d}{dx} 4(\ln(2x+1) - \ln(3x-7))$

$= 4\left(\frac{1}{2x+1}\cdot 2 - \frac{1}{3x-7}\cdot 3\right) = \frac{8}{2x+1} - \frac{12}{3x-7}$

5. (5pts) Use logarithmic differentiation to find $\frac{d}{dx}(x^2+3x-1)^{\sin x}$.

$y = (x^2+3x-1)^{\sin x} \quad | \ln$

$\ln y = \ln(x^2+3x-1)^{\sin x}$

$\ln y = \sin x \ln(x^2+3x-1) \quad | \frac{d}{dx}$

$\frac{y'}{y} = \cos x \ln(x^2+3x-1) + \sin x \frac{1}{x^2+3x-1} \cdot (2x+3) \quad | \cdot y$

$y = \left(\cos x \ln(x^2+3x-1) + \frac{(2x+3)\sin x}{x^2+3x-1}\right)(x^2+3x-1)^{\sin x}$

6. (4pts) Find the equation of the tangent line to the curve $y = x^3 - 4x^2 + 7$ at the point $(1, 4)$.

$y' = 3x^2 - 8x$

$y'(1) = 3 - 8 = -5$

slope of tan. line

$y - 4 = -5(x-1)$

$y = -5x + 9$

7. (4pts) Find the first three derivatives of $f(x)$ and use them to find the formula for $f^{(n)}(x)$ if $f(x) = \ln x$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = (-1)x^{-2}$

$f'''(x) = (-1)(-2)x^{-3}$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$

$f^{(n)}(x) = (-1)^{n-1} 1 \cdot 2 \cdot \ldots \cdot (n-1) x^{-n}$

$= (-1)^{n-1}(n-1)!\, x^{-n}$

8. (5pts) Use implicit differentiation to find y'.

$\tan(xy) = 3x^2 + 5y^4$ $\left| \frac{d}{dx} \right.$

$\sec^2(xy)\,(xy)' = 6x + 20y^3 y'$

$\sec^2 xy\,(y + xy') = 6x + 20y^3 y'$

$y\sec^2(xy) - 6x = 20y^3 y' - x\sec^2(xy) \cdot y'$

$y\sec^2(xy) - 6x = \left(20y^3 - x\sec^2(xy)\right) y'$

$$y' = \frac{y\sec^2(xy) - 6x}{20y^3 - x\sec^2(xy)}$$

9. (8pts) A tank filled with 600 liters of water drains in 4 hours from an opening in the bottom. The volume of water in the tank after t hours is given by $V(t) = 600(1 - \frac{t}{4})^2$.
a) How much water is in the tank when $t = 2$?
b) At what rate is the water draining when $t = 2$? What are the units?
c) Interpret the meaning of the number in b) by approximating how much water there is in the tank at time $t = 2.1$.
d) What is the exact amount of water in the tank at time $t = 2.1$?

a) $V(2) = 600\left(1 - \frac{2}{4}\right)^2 = 600 \cdot \frac{1}{4} = 150$ liters

b) $V'(t) = 600 \cdot 2 \cdot \left(1 - \frac{t}{4}\right)\left(-\frac{1}{4}\right)$

$= -\frac{1200}{4}\left(1 - \frac{t}{4}\right)$

$= -300\left(1 - \frac{t}{4}\right)$

$V'(2) = -150$ liters per hour

c) $V'(2) = -150$ means that in 0.1 hrs the amount of water will drop approximately by $150 \cdot 0.1 = 15$ liters

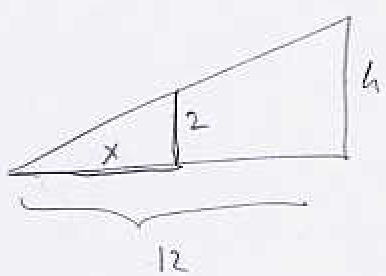
thus $V(2.1) \approx 150 - 15 = 135$ liters

d) $V(2.1) = 600\left(1 - \frac{2.1}{4}\right)^2 = 600(0.475)^2 = 135.375$ liters ← close

10. (7pts) A spotlight on the ground shines on a wall 12 meters away. If a man 2 meters tall walks from the spotlight to the wall at a speed of 1.6 meters per second, how fast is the length of his shadow on the wall decreasing when he is 8 meters away from the spotlight?

Know: $x' = 1.6$

Need: h', when $x = 8$.



$$\frac{2}{x} = \frac{h}{12}$$

$$h = \frac{24}{x} \quad \Big| \frac{d}{dt}$$

$$h'(t) = -24x^{-2} \cdot x' = -\frac{24x'}{x^2}$$

when $x = 8$ we have

$$h' = -\frac{24 \cdot 1.6}{8^2} = -0.6 \text{ m/s}$$

Bonus. (5pts) Let $h(x) = f(x)g(x)$. Find the formula for $h''(x)$ in terms of f, f', f'', g, g', g''. What familiar formula from algebra does it resemble?

$$h = fg \quad \Big| \frac{d}{dx}$$

$$h' = f'g + fg' \quad \Big| \frac{d}{dx}$$

$$h'' = f''g + f'g' + f'g' + fg''$$

$$= f''g + 2f'g' + fg''$$

If we write $g = g^{(0)}$ (0 times differentiated)

this is

$$(fg)^{(2)} = f^{(2)}g^{(0)} + 2f^{(1)}g^{(1)} + f^{(0)}g^{(2)}$$

similar to

$$(a+b)^2 = a^2b^0 + 2a^1b^1 + a b^2$$