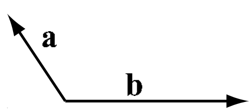
Fall '07/MAT 309/Exam 1 Name: *Show all your work.*

1. (6pts) Find $\langle 3, 2+t, 5-2t\rangle \times \langle 2, 1, t+1\rangle =$

2. (17pts) Let $\mathbf{a}$ and $\mathbf{b}$ be vectors sketched below.
a) draw the vectors $-3\mathbf{a}$, $\mathbf{a}-\mathbf{b}$ and $2\mathbf{a}+\mathbf{b}$.
b) draw the vector $\text{proj}_{\mathbf{b}}\,\mathbf{a}$.
c) If $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i}$, find the coordinates of $\text{proj}_{\mathbf{b}}\,\mathbf{a}$.



3. (6pts) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors, and u, v scalars. Are the following expressions defined? For those that are not, explain what is wrong.

$\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$ $(\mathbf{b} \cdot \mathbf{c})\mathbf{a} \times \mathbf{c}$

4. (10pts) The spherical coordinates of a point are $\left(4, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}\right)$. Sketch the point and find its cylindrical coordinates (exact numbers, not decimal).

5. (9pts) In cylindrical coordinates, draw the solid described by:

$3 \leq r \leq 5$, $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$, $0 \leq z \leq 4$

6. (13pts) This problem is about the surface $y = \dfrac{x^2}{9} + \dfrac{z^2}{16}$.
a) Sketch and identify the intersections of this surface with the coordinate planes.
b) Sketch the surface in 3D, with coordinate system visible.

7. (9pts) The parametric equations of a curve are $x = 3\cos t$, $y = t^2$, $z = 3\sin t$, $0 \leq t \leq 6\pi$. Sketch this curve.

8. (17pts) A jet-powered eggplant travels along the path $x = t^2 + 2t$, $y = \dfrac{3}{t+1}$, $z = t^2 e^t$. At the point $(8, 1, 4e^2)$ it experiences engine failure, so from this point on, it continues along the tangent line to this curve and splatters on the xz-plane.

a) Find the parametric equations of the line tangent to the curve at $(8, 1, 4e^2)$.

b) At which point does the tangent line intersect the xz-plane? (This is where the eggplant splatters if gravitation is ignored.)

9. (13pts) Find the equation of the plane that contains the line $\dfrac{x-2}{1} = \dfrac{y+4}{-3} = \dfrac{z-1}{2}$ and the point $(3, 0, 4)$.

Bonus. (10pts) This problem is about the vector $\mathbf{d} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

a) Explain why $\mathbf{d}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$.

b) Since $\mathbf{d}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, it can be written as $\mathbf{d} = u\mathbf{b} + v\mathbf{c}$, for some scalars u and v. Find a relationship between u and v by dotting the equation $\mathbf{d} = u\mathbf{b} + v\mathbf{c}$ by $\mathbf{a}$.

1. (13pts) Let $f(x,y) = \sqrt{x^2 + y^2 - 4}$.
a) Find the domain of f and sketch it.
b) Find the level curves for the levels $k = -1, 0, 1, 2$ and draw them.
c) The point $(1,2)$ is on the level curve for $k = 1$. Use b) to roughly draw the vector $\nabla f(1,2)$ emanating from the point $(1,2)$.

2. (14pts) At a cheese factory in Muscoda, WI, the intensity of the cheese smell is given by $L(x,y) = 4 + x^3 + y^3 - 3xy$.
a) Susan stands at point $(3,0)$ and sees Joe at point $(1,1)$. If she starts moving toward him, will the cheese smell intensify? At what rate?
b) In which direction should Susan move to experience the maximum drop in smell? At which rate will the smell intensity drop?

3. (9pts) A rectangular box is measured to have length $x = 30$cm, width $y = 16$cm and height $z = 90$cm, with an error in measurement at most 0.5cm in each. Use differentials to estimate the maximal error in computing the volume of the box.

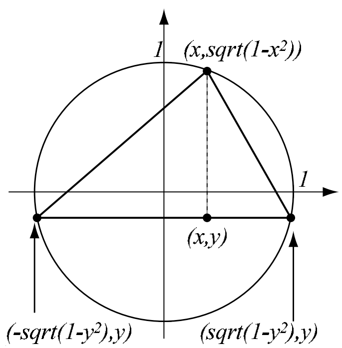
4. (16pts) Let $z = \dfrac{e^{xy}}{y}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 4$ and $\theta = \frac{\pi}{2}$.

5. (15pts) The equation $xz^3 + y\ln(x + y^2 z) = 7$ defines z implicitly as a function of x and y. Let $F(x, y, z) = xz^3 + y\ln(x + y^2 z) - 7$. Find F_x, F_y and F_z and use them to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. (Do not simplify.)

6. (15pts) A circle of radius 3 is parametrized by $x = 3\cos t$, $y = 3\sin t$.
a) Find the circumference of the circle using the formula for length of parametric curves.
b) Reparametrize the curve with respect to arc length, measured from the point for which $t = 0$.

7. (18pts) Find and classify the local extremes for the function $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$.

Bonus. (12pts) Among all triangles inscribed in the unit circle, find the one with the largest area. Tips:
1) By rotating, the triangle can be positioned so that one of its heights is vertical and the vertex on the height is on the upper half-circle, like in the picture.
2) Write the area $A(x, y)$ of the triangle in terms of x and y from the picture. Your job is to maximize $A(x, y)$ over a certain closed region in $\mathbf{R}^2$. What is the region? (Note: the triangle is completely determined by the point (x, y) where the height meets the base.)
3) What are the critical points of $A(x, y)$? (Squaring A won't help much. Find A_x and A_y directly.) Then investigate values on the boundary of the region.



1. (16pts) Find $\iint_D y^2 \, dA$ if D is the region bounded by the lines $y = 0$, $y = -x$ and $y = \frac{1}{2} - \frac{x}{2}$. Sketch the region of integration.

2. (16pts) Evaluate $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \sqrt{1+y^3} \, dy \, dx$ by changing the order of integration. Sketch the region of integration.

3. (10pts) Set up $\iint_D x\,dA$ in polar coordinates if D is the region inside the first-quadrant petal of the curve $r = \sin 2\theta$ that is also above the line $y = x$. Sketch the region, but do not evaluate the integral.

4. (12pts) Sketch the region whose volume is given by the triple integral below:

$$\int_0^1 \int_{-\sqrt{2-x}}^{\sqrt{2-x}} \int_0^{4-4y} 1\,dz\,dy\,dx$$

5. (16pts) Use cylindrical coordinates to set up $\iiint_E xyz^2\,dV$ where E is the region above the paraboloid $z = \frac{1}{2}(x^2 + y^2)$, under the sphere $x^2 + y^2 + z^2 = 35$ and between the planes $y = \sqrt{3}x$ and $y = -\sqrt{3}x$. Sketch the region of integration. Do not evaluate the integral.

6. (16pts) Sketch the region E bounded by the planes $z = 0$, $x = 0$, $2x + y + z = 6$ and $y - 2z = 0$. Then write the iterated triple integral that stands for $\iiint_E f\,dV$ that ends in $dy\,dz\,dx$.

7. (14pts) Use change of variables to find the area of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{25} = 1$.

Bonus. (10pts) Consider ther region below the paraboloid $z = \frac{1}{2}(x^2 + y^2)$, inside the sphere $x^2 + y^2 + z^2 = 35$, and above the xy-plane.
a) Set up the triple integral for the volume of this region in spherical coordinates.
b) Evaluate the integral, with final answer in exact form (not decimal!).

1. (10pts) Roughly draw the vector field ∇f if $f(x,y) = y - x^2$. Note that it is possible to do this with no computation.

2. (10pts) Write the parametric equations for the part of the plane $z = 4 - y$ that lies inside of the cylinder $x^2 + z^2 = 10$. Specify the planar region D where your parameters come from.

3. (25pts) Let $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + e^y \sin z)\,\mathbf{j} + e^y \cos z\,\mathbf{k}$ be a vector field.
a) Find curl $\mathbf{F}$.
b) Is the field $\mathbf{F}$ conservative? If it is, find the function f so that $\nabla f = \mathbf{F}$.
c) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is any curve from $(2, -1, 0)$ to $(3, 1, \frac{\pi}{2})$.

4. (10pts) Let $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$. Show that $\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div} \mathbf{F}$.

5. (25pts) Let D be the region inside the unit circle, and let C be its boundary, oriented clockwise. Evaluate the integral $\int_C xy^2\, dx + yx^2\, dy$ in two ways:

a) directly

b) using Green's theorem.

If you don't get the same answer in a) and b), write "-5" on the margin. (Just kidding! Go to next problem and then check back.)

6. (20pts) Set up the double integral for $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where the surface S is the part of the plane $x+y+2z=6$ in the first octant, and $\mathbf{F}(x,y) = \langle -y, x, x+y+z\rangle$. Use the downward-pointing normal vector. Carry out the set-up until you get iterated single integrals, but do not evaluate the integral.

Bonus (10pts) Find the surface area of the part of the sphere $x^2+y^2+z^2=R^2$ that lies above the plane $z=h$, $0 \le h \le R$.

1. (7pts) Find the spherical coordinates of the point whose rectangular coordinates are $(-1, 2, 5)$.

2. (12pts) Let $z = x\sin(xy)$, $x = 8t - t^4$, $y = e^{\frac{1}{t}}$. Find $\dfrac{dz}{dt}$ when $t = 2$.

3. (10pts) Find parametric equations of the line that is the intersection of the planes $2x + y - 3z = 5$ and $x - y + 2z = 4$.

4. (10pts) Let $f(x, y) = y - x^3$.
a) Draw the level curves for f for the levels $k = -2, -1, 0, 1, 2$.
b) Roughly draw the vector field ∇f. Note that no computation is needed for this.
c) Compute $\int_C \nabla f \cdot d\mathbf{r}$, where C is the vertical line segment joining points $(1, -3)$ and $(1, 4)$.
d) If you are standing at the point $(4, -3)$, in which direction should you move to experience the greatest increase in f?

5. (15pts) Find and classify the local extremes for the function $f(x, y) = x^4 + y^4 - 4xy + 2$.

6. (16pts) Find $\iint_D \sin y^2 \, dA$ if D is the region bounded by the graph of $y = |x|$ and the line $y = 4$. Sketch the region of integration.

7. (14pts) Use either spherical or cylindrical coordinates to set up $\iiint_E z^2(x^2+y^2)\,dV$, where E is the region above the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and under the sphere $x^2+y^2+z^2 = 16$. Sketch the region of integration. Do not evaluate the integral.

8. (14pts) Sketch the region E bounded by the planes $z = 3$, $y = 0$, $z = 2x$ and the surface $y = \sqrt{x}$. Then write the iterated triple integral that stands for $\iiint_E f\,dV$ that ends in $dx\,dz\,dy$.

9. (22pts) Let D be the region between the curve $y = 4 - x^2$ and the x-axis and let C be its boundary, oriented in the positive (counterclockwise) direction.
a) Set up the two integrals needed to find $\int_C xy^2\,dx + 2x^2y\,dy$ and evaluate the easy one.
b) Find $\int_C xy^2\,dx + 2x^2y\,dy$ using Green's theorem.

10. (20pts) Let S be the part of the cylinder $y^2 + z^2 = 9$ that is between planes $x = 0$ and $x = 5$. Choose normal vectors for S so that they point away from the x-axis.
a) Write the parametric equations for this surface. Specify the planar region D where your parameters come from.
b) Use your parametrization to set up the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$.
c) Evaluate the integral from b).

Bonus (14pts) This problem is about the surface $x^2 + y^2 - z^2 = 1$.
a) Sketch and identify the intersections of this surface with the plane $z = k$.
b) Sketch the intersection of this surface with the rz-plane.
c) Use a) and b) to sketch the surface in 3D, with coordinate system visible.