Fall '07/MAT 309/Final Exam **Name:** Solution *Show all your work.*

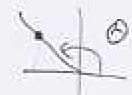
1. (7pts) Find the spherical coordinates of the point whose rectangular coordinates are $(-1, 2, 5)$.

$\rho^2 = (-1)^2 + 2^2 + 5^2$

$= 1 + 4 + 25 = 30$

$\rho = \sqrt{30}$

$\tan\theta = \frac{y}{x} = \frac{2}{-1}$



$\theta = \underbrace{\arctan(-2) + \pi}_{\text{to get it in 2nd quadrant}} \approx 2.0344$ rad

(arctan -2 is in quad. 4)

$\rho\cos\phi = z$

$\sqrt{30}\cos\phi = 5$

$\phi = \arccos\frac{5}{\sqrt{30}} \approx 0.4205$ rad

2. (12pts) Let $z = x\sin(xy)$, $x = 8t - t^4$, $y = e^{\frac{1}{t}}$. Find $\frac{dz}{dt}$ when $t = 2$.

$\frac{\partial z}{\partial x} = \sin(xy) + x\cos(xy)\cdot y$

$= \sin(xy) + xy\cos(xy)$

$t = 2 \Rightarrow x = 0$

$y = \sqrt{e}$

$\frac{\partial z}{\partial y} = x\cos(xy)\cdot x = x^2\cos(xy)$

$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$

$= \left(\sin(xy) + xy\cos(xy)\right)(8 - 4t^3) + x^2\cos(xy)\left(e^{\frac{1}{t}}\left(-\frac{1}{t^2}\right)\right)$

Put in $t = 2$ $\begin{pmatrix} x = 0 \\ y = \sqrt{e} \end{pmatrix}$ $= 0\cdot(8 - 4\cdot 2^3) + 0\cdot 1\cdot\frac{\sqrt{e}}{-4} = 0$

3. (10pts) Find parametric equations of the line that is the intersection of the planes $2x + y - 3z = 5$ and $x - y + 2z = 4$.

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & -1 & 2 \end{vmatrix} = \langle -1, -7, -3 \rangle$$

A pt. on the intersection: set $z=0$

$$\begin{aligned} 2x+y&=5 \\ x-y&=4 \\ \hline 3x&=9 \end{aligned} \Rightarrow \begin{aligned} x&=3 \\ y&=-1 \\ z&=0 \end{aligned}$$

Parametric equations of a line through $(3,-1,0)$ with direction vector $\langle 1,7,3\rangle$:

$$x = 3+t$$
$$y = -1+7t$$
$$z = 3t$$

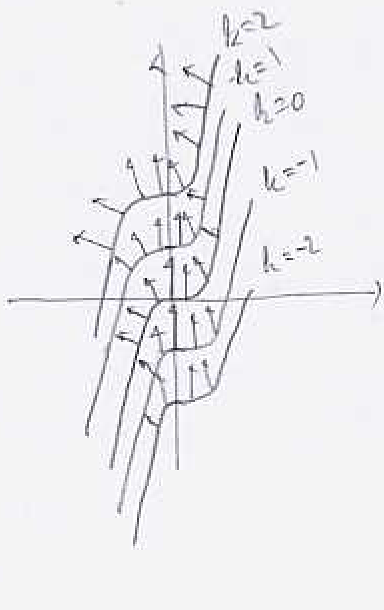
4. (10pts) Let $f(x, y) = y - x^3$.
a) Draw the level curves for f for the levels $k = -2, -1, 0, 1, 2$.
b) Roughly draw the vector field ∇f. Note that no computation is needed for this.
c) Compute $\int_C \nabla f \cdot d\mathbf{r}$, where C is the vertical line segment joining points $(1, -3)$ and $(1, 4)$.
d) If you are standing at the point $(4, -3)$, in which direction should you move to experience the greatest increase in f?

a) $f(x,y)=k$

$y-x^3=k$

$y=x^3+k$

b) ∇f is normal to level curves, points in direction of increasing k.



c) $\int \nabla f \cdot dr = f(1,4) - f(1,-3)$
$= (4-1^3) - (-3-1^3)$
$= 7$

d) need to move in the direction of $\nabla f(4,-3) = \langle -48, 1\rangle$

$\nabla f = \langle -3x^2, 1\rangle$

5. (15pts) Find and classify the local extremes for the function $f(x, y) = x^4 + y^4 - 4xy + 2$.

$\frac{\partial f}{\partial x} = 4x^3 - 4y$

$\frac{\partial f}{\partial y} = 4y^3 - 4x$

$D = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16$

$\left.\begin{array}{l} 4x^3 - 4y = 0 \Rightarrow x^3 = y \\ 4y^3 - 4x = 0 \Rightarrow y^3 = x \end{array}\right\} \Rightarrow x^9 = x$

$x(x^8 - 1) = 0$

$x = 0$ or $x^8 = 1$

$\Downarrow$ $\quad$ $x = \pm 1$

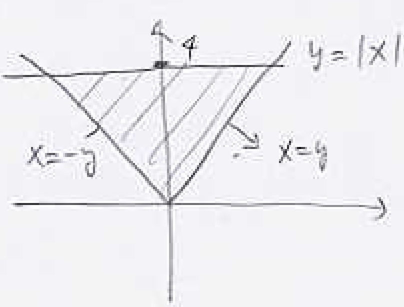
$y = 0$ $\quad$ $\Downarrow$

$y = \pm 1$

Crit. pts	$D(x,y)$
$(0,0)$	-16 so saddle point
$(1,1)$	128, since $12x^2 > 0$ it is a local min
$(-1,-1)$	128, — ———

Critical pts: $(0,0)$, $(1,1)$, $(-1,-1)$

6. (16pts) Find $\iint_D \sin y^2 \, dA$ if D is the region bounded by the graph of $y = |x|$ and the line $y = 4$. Sketch the region of integration.

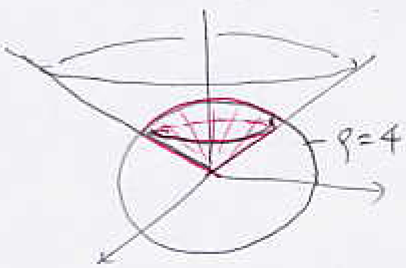


$\int_0^4 \int_{-y}^{y} \sin y^2 \, dx \, dy$

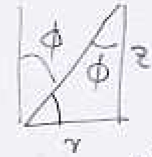
$= \int_0^4 \sin y^2 \cdot 2y \, dy = \left[\begin{array}{ll} u = y^2 & y = 4, u = 16 \\ du = 2y\,dy & y = 0, u = 0 \end{array}\right]$

$= \int_0^{16} \sin u \, du = -\cos u \Big|_0^{16} = 1 - \cos 16$

7. (14pts) Use either spherical or cylindrical coordinates to set up $\iiint_E z^2(x^2+y^2)\,dV$, where E is the region above the cone $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and under the sphere $x^2+y^2+z^2 = 16$. Sketch the region of integration. Do not evaluate the integral.



$z = \frac{1}{\sqrt{3}} r$ $\rho\cos\phi = \frac{1}{\sqrt{3}}\rho\sin\phi$

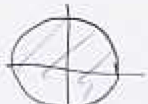


$r = \rho\sin\phi$ $\tan\phi = \sqrt{3}$

$z = \rho\cos\phi$ $\phi = \arctan\sqrt{3} = \frac{\pi}{3}$ $\tan\phi = \frac{r}{z}$

Spherical: $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^2\cos^2\phi\,(r^2)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ $r^2 = \rho^2\sin^2\phi$

Proj. to xy-plane



$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^4 \rho^6\cos^2\phi\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$z = \frac{1}{\sqrt{3}} r$

$r^2 + z^2 = 16$ $r^2 = 12$

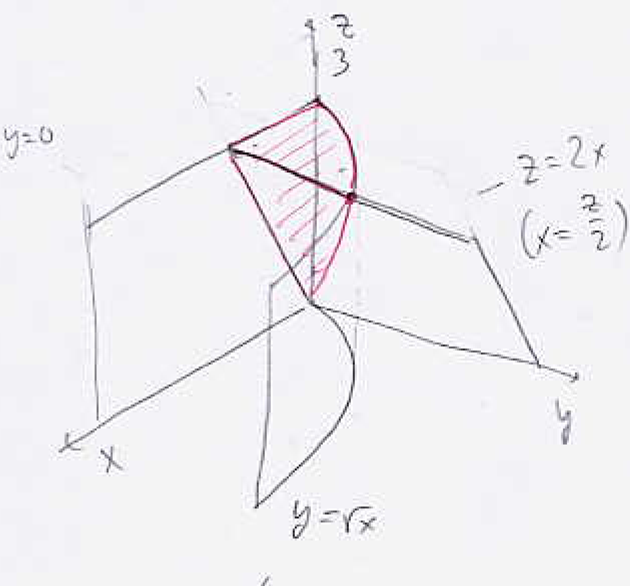
$r^2 + \frac{r^2}{3} = 16$ $r = \sqrt{12}$

$\frac{4}{3} r^2 = 16$

Cylindrical: $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{12}}\int_{\frac{1}{\sqrt{3}}r}^{\sqrt{16-r^2}} z^2 r^2\cdot r\,dz\,dr\,d\theta$

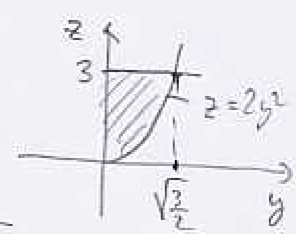
$$= \int_0^{2\pi}\int_0^{\sqrt{12}}\int_{\frac{1}{\sqrt{3}}r}^{\sqrt{16-r^2}} z^2 r^3\,dz\,dr\,d\theta$$

8. (14pts) Sketch the region E bounded by the planes $z = 3$, $y = 0$, $z = 2x$ and the surface $y = \sqrt{x}$. Then write the iterated triple integral that stands for $\iiint_E f\,dV$ that ends in $dx\,dz\,dy$.



Proj. to yz-plane

$\begin{cases} z = 2x \\ y = \sqrt{x} \end{cases} \Rightarrow z = 2y^2$



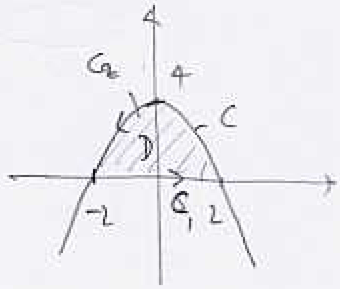
$3 = 2y^2, \Rightarrow y = \sqrt{\frac{3}{2}}$

$$\int_0^{\sqrt{3/2}}\int_{2y^2}^{3}\int_{y^2}^{\frac{z}{2}} f\,dx\,dz\,dy$$

$\frac{\partial Q}{\partial x} = 2y^2 \quad \frac{\partial P}{\partial y} = x^2$

9. (22pts) Let D be the region between the curve $y = 4 - x^2$ and the x-axis and let C be its boundary, oriented in the positive (counterclockwise) direction.
a) Set up the two integrals needed to find $\int_C x^4 y^2 dx + 2x^2 y^5 dy$ and evaluate the easy one.
b) Find $\int_C x^4 y^2 dx + 2x^2 y^4 dy$ using Green's theorem.

a)



C consists of 2 parts:

C_1: $x = t \quad dx = 1 \quad t \in [-2, 2]$
$\quad y = 0 \quad dy = 0$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-2}^{2} t \cdot 0 \cdot 1 + 2 \cdot t^2 0 \cdot 0 \, dt = 0$$

$-C_2$: $x = t \quad dx = 1 \quad t \in [-2, 2]$
$\quad y = 4 - t^2 \quad dy = -2t$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-2}^{2} t(4-t^2)^2 \cdot 1 + 2 \cdot t^2 (4-t^2) \cdot (-2t) \, dt$$

$$= \int_{-2}^{2} t(4-t^2)(4-t^2-4t^2) \, dt$$

$$= \int_{-2}^{2} t(4-t^2)(4-5t^2) \, dt = 0$$

b/c it is an odd function, integrated over a symm. interval.

b) $\int_C xy^2 dx + 2x^2 y \, dy$

$$= \iint_D \frac{\partial}{\partial x}(2x^2 y) - \frac{\partial}{\partial y}(xy^2) \, dA$$

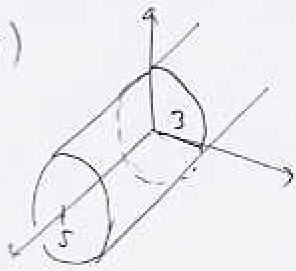
$$= \iint_D 4xy - 2xy \, dA = \iint_D 2xy \, dA = 2 \int_{-2}^{2} \int_{0}^{4-x^2} xy \, dy \, dx$$

$$= \int_{-2}^{2} x(4-x^2)^2 \, dx = \begin{bmatrix} u = 4 - x^2 & x = 2, u = 0 \\ du = -2x \, dx & x = -2, u = 0 \end{bmatrix} = \int_0^0 -\frac{u^2}{2} du = 0$$

$-\frac{du}{2} = x \, dx$

10. (20pts) Let S be the part of the cylinder $y^2 + z^2 = 9$ that is between planes $x = 0$ and $x = 5$. Choose normal vectors for S so that they point away from the x-axis.
a) Write the parametric equations for this surface. Specify the planar region D where your parameters come from.
b) Use your parametrization to set up the integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.
c) Evaluate the integral from b).

a)



$y = 3\cos\theta \qquad 0 \le \theta \le 2\pi$

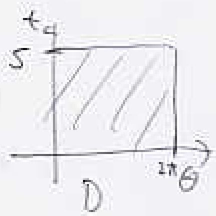
$z = 3\sin\theta \qquad 0 \le t \le 5$

$x = t$

$x = t$

$y = 3\cos\theta$

$z = 3\sin\theta$



b) $\vec{r}_\theta \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -3\sin\theta & 3\cos\theta \\ 1 & 0 & 0 \end{vmatrix} = \underbrace{\langle 0, 3\cos\theta, 3\sin\theta \rangle}_{\text{points away from } x\text{-axis}}$

$$\iint_D \langle t, 3\cos\theta, 3\sin\theta \rangle \cdot \langle 0, 3\cos\theta, 3\sin\theta \rangle \, dA$$

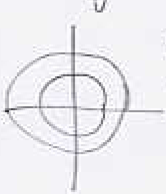
$$= \iint_D 9\cos^2\theta + 9\sin^2\theta \, dA = \iint_D 9 \, dA$$

c) $= 9 \cdot \text{area of } D = 9 \cdot 2\pi \cdot 5 = 90\pi$

Bonus (14pts) This problem is about the surface $x^2 + y^2 - z^2 = 1$.
a) Sketch and identify the intersections of this surface with the plane $z = k$.
b) Sketch the intersection of this surface with the rz-plane.
c) Use a) and b) to sketch the surface in 3D, with coordinate system visible.

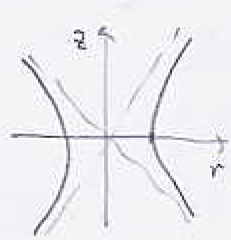
a) $z = k$

$x^2 + y^2 = 1 + k^2$



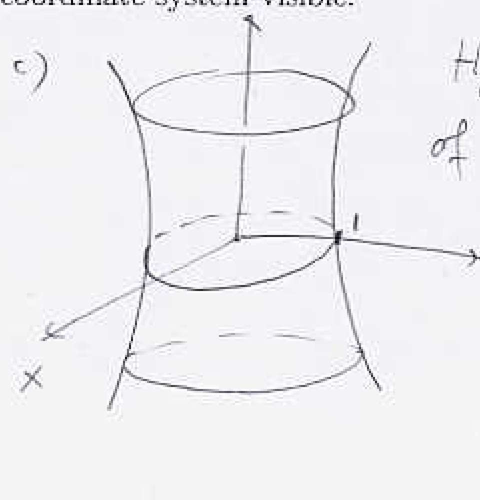
circles of radius $\sqrt{1+k^2}$

b) $r^2 - z^2 = 1$



may rotate about z axis

c)



Hyperboloid of one sheet