Fall '07/MAT 309/Exam 4 **Name:** Solution *Show all your work.*

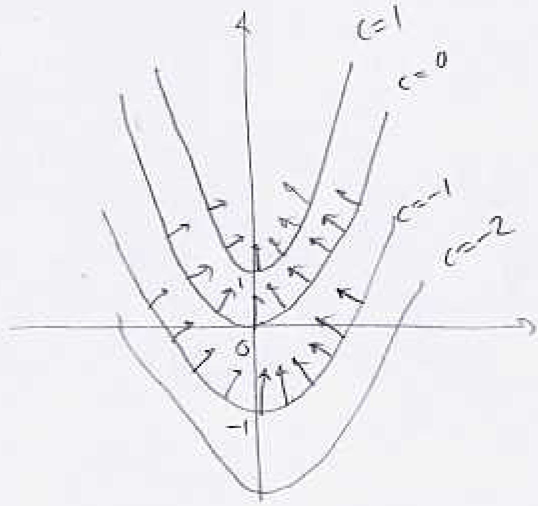
1. (10pts) Roughly draw the vector field ∇f if $f(x, y) = y - x^2$. Note that it is possible to do this with no computation.

∇f is perpendicular to level curves of f

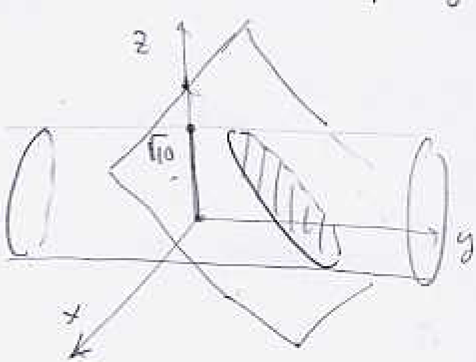
$y - x^2 = c$

$y = x^2 + c$

parabolas

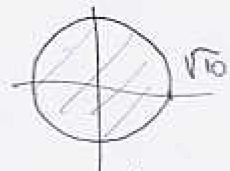


2. (10pts) Write the parametric equations for the part of the plane $z = 4 - y$ that lies inside of the cylinder $x^2 + z^2 = 10$. Specify D



$z = 4 - y$

Surface projects to disk



in the xz-plane

May use

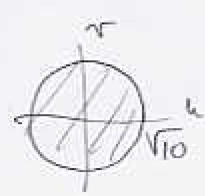
$y = 4 - z$

$x = u$

$y = 4 - v$

$z = v$

D =



3. (20pts) Let $\mathbf{F}(x,y,z) = 2xy\,\mathbf{i} + (x^2 + e^y \sin z)\,\mathbf{j} + e^y \cos z\,\mathbf{k}$ be a vector field.
a) Is this field conservative? If it is, find the function f so that $\nabla f = \mathbf{F}$.
b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, if C is any curve from $(2,-1,0)$ to $(3,1,\frac{\pi}{2})$.

a) $\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2 + e^y\sin z & e^y\cos z \end{vmatrix} = \langle e^y\cos z - e^y\cos z,\ 0-0,\ 2x-2x \rangle = \vec{0}$

Since $\vec{F}$ is defined on all of $\mathbb{R}^3$ and $\operatorname{curl}\vec{F} = \vec{0}$, $\vec{F}$ is conservative

$\frac{\partial f}{\partial x} = 2xy \quad \int dx$

$f = x^2y + g(y,z)$

$x^2 + e^y\sin z = \frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y}(y,z)$

$\Rightarrow \frac{\partial g}{\partial y} = e^y\sin z \quad \Big| \int dy$

$g = e^y\sin z + h(z)$

$f = x^2y + e^y\sin z + h(z)$

$e^y\cos z = \frac{\partial f}{\partial z} = e^y\cos z + h'(z)$

$h'(z) = 0$,

$h(z) = k$

$f = x^2y + e^y\sin z$

b) $\int_C \vec{F}\cdot d\vec{r} = f(3,1,\frac{\pi}{2}) - f(2,-1,0) = (9 + e^1\cdot 1) - (4(-1)+0) = 13 + e$

4. (10pts) Let $\mathbf{F}(x,y,z) = \langle P, Q, R \rangle$. Show that $\operatorname{div}(f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f \operatorname{div}\mathbf{F}$.

$\operatorname{div}(f\vec{F}) = \operatorname{div}\langle fP, fQ, fR \rangle$

$= \frac{\partial}{\partial x}(fP) + \frac{\partial}{\partial y}(fQ) + \frac{\partial}{\partial z}(fR)$

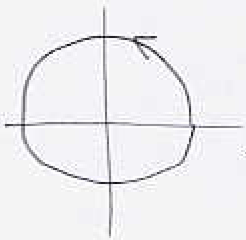
$= \frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} + \frac{\partial f}{\partial y}\cdot Q + f\cdot\frac{\partial Q}{\partial y} + \frac{\partial f}{\partial z}\cdot R + f\frac{\partial R}{\partial z}$

$= \frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q + \frac{\partial f}{\partial z}R + f\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$

$= \nabla f\cdot\vec{F} + f\cdot\operatorname{div}\vec{F}$

5. (25pts) Let D be the region inside the unit circle, and let C be its boundary, oriented clockwise. Evaluate the integral $\int_C xy^2\,dx + yx^2\,dy$ in two ways:
a) directly
b) using Green's theorem.

a)



$x = \cos t \quad x' = -\sin t$
$y = \sin t \quad y' = \cos t$
$t \in [0, 2\pi)$

$$\int_0^{2\pi} \underbrace{\cos t}_{x}\, \underbrace{\sin^2 t}_{y^2}\, \underbrace{(-\sin t)}_{dx} + \sin t\,(\cos t)^2 \cos t \, dt$$

$$= \int_0^{2\pi} \sin t \cos^3 t - \cos t \sin^3 t \, dt$$

$$= \int_0^{2\pi} \sin t \cos^3 t \, dt - \int_0^{2\pi} \cos t \sin^3 t \, dt$$

$$= \left[\begin{array}{ll|ll} u = \cos t & t = 2\pi, u = 1 & v = \sin t & t = 2\pi, v = 0 \\ du = -\sin t\,dt & t = 0, u = 1 & dv = \cos t\,dt & t = 0, v = 0 \end{array} \right]$$

$$= \int_1^1 -u^3\,du + \int_0^0 u^3\,du = 0 + 0 = 0$$

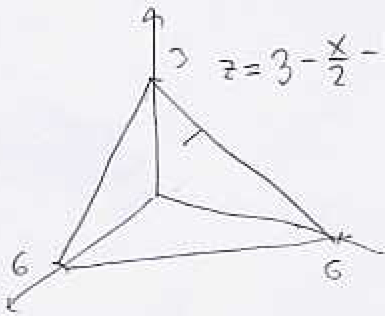
b) $$\iint_D \frac{\partial}{\partial x}(yx^2) - \frac{\partial}{\partial y}(xy^2)\,dA$$

$$= \iint_D 2xy - 2xy \, dA$$

$$= 0$$

If you don't get the same answer in a) and b), write "-5" on the margin. (Just kidding! Go to next problem and then check back.)

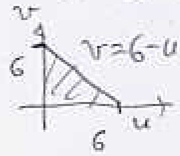
6. (20pts) Set up the double integral for $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where the surface S is the part of the plane $x+y+2z=6$ in the first octant, and $\mathbf{F}(x,y) = \langle -y, x, x+y+z\rangle$. Use the downward-pointing normal vector. Carry out the set-up until you get iterated single integrals, but do not evaluate the integral.



Param; $x = u$
$y = v$
$z = 3 - \frac{u}{2} - \frac{v}{2}$

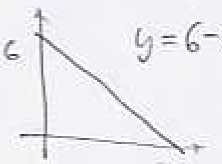
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \langle \tfrac{1}{2}, \tfrac{1}{2}, 1\rangle$$

D =



Downward-pointing is $\langle -\frac{1}{2}, -\frac{1}{2}, -1\rangle$

proj. to xy-plane

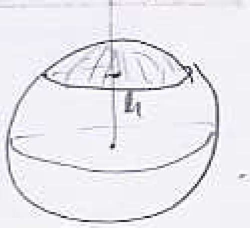


$$\iint_D \langle -v, u, u+v+3-\tfrac{u}{2}-\tfrac{v}{2}\rangle \cdot \langle -\tfrac{1}{2}, -\tfrac{1}{2}, -1\rangle \, dA$$

$$= \iint_D \frac{v}{2} - \frac{u}{2} - u - v - 3 + \frac{u}{2} + \frac{v}{2} \, dA$$

$$= \int_0^6 \int_0^{6-u} -u-3 \, dv \, du$$

Bonus (10pts) Find the surface area of the part of the sphere $x^2+y^2+z^2=R^2$ that lies above the plane $z=h$, $0 \le h \le R$.

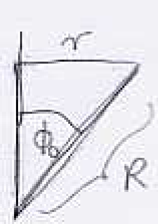


Parametrization (uses spherical coordinates;)

$x = R\sin\phi\cos\theta$ $\quad 0 \le \theta \le \frac{\pi}{2}$

$y = R\sin\phi\sin\theta$ $\quad 0 \le \phi \le \phi_0$

$z = R\cos\phi$



$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -R\sin\phi\sin\theta & R\sin\phi\cos\theta & 0 \\ -R\cos\phi\cos\theta & R\cos\phi\sin\theta & -R\sin\phi \end{vmatrix} = \langle -R^2\sin^2\phi\cos\theta, -R^2\sin^2\phi\sin\theta, R^2\sin\phi\cos\phi\rangle$$

$\cos\phi_0 = \frac{h}{R}$

$\phi_0 = \arccos\frac{h}{R}$

$$|\vec{r}_\theta \times \vec{r}_\phi| = \sqrt{R^4\sin^4\phi\cos^2\theta + R^4\sin^4\phi\sin^2\theta + R^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{R^4\sin^4\phi + R^4\sin^2\phi\cos^2\phi} = R^2\sin\phi$$

$$A = \int_0^{2\pi}\int_0^{\phi_0} R^2\sin\phi \, d\phi \, d\theta = 2\pi R^2(-\cos\phi)\Big|_0^{\phi_0} = 2\pi R^2(1-\cos\phi_0)$$

$$= 2\pi R^2\left(1-\frac{h}{R}\right) = 2\pi R(R-h)$$