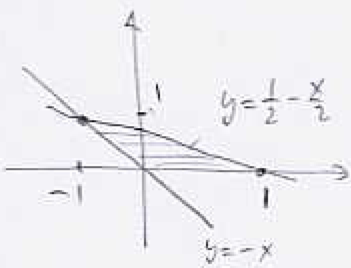
Fall '07/MAT 309/Exam 3 Name: Solution *Show all your work.*

1. (14pts) 6 Find $\iint_D y^2\,dA$ if D is the region bounded by the lines $y=0$, $y=-x$ and $y=\frac{1}{2}-\frac{x}{2}$. Sketch the region of integration.



$$\iint_D y^2\,dA = \int_0^1\int_{-y}^{1-2y} y^2\,dx\,dy$$

type 2

$$= \int_0^1 y^2\big(1-2y-(-y)\big)\,dy = \int_0^1 y^2(1-y)\,dy$$

$$= \int_0^1 y^2-y^3\,dy = \frac{y^3}{3}-\frac{y^4}{4}\Big|_0^1 = \frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

$\frac{1}{2}-\frac{x}{2}=-x$

$\frac{1}{2}x=-\frac{1}{2}$

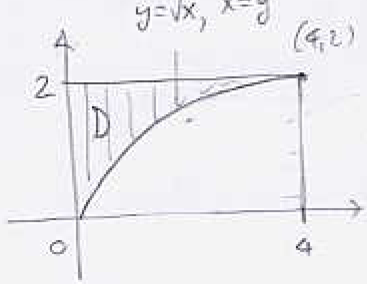
$x=-1$

$y=\frac{1}{2}-\frac{x}{2}$

$x=1-2y$

2. (14pts) 6 Evaluate $\int_0^4\int_{\sqrt{x}}^2 \sqrt{1+y^3}\,dy\,dx$ by changing the order of integration. Sketch the region of integration.

$y=\sqrt{x}$, $x=y^2$



$$\iint_D \sqrt{1+y^3}\,dA = \int_0^2\int_0^{y^2}\sqrt{1+y^3}\,dx\,dy$$

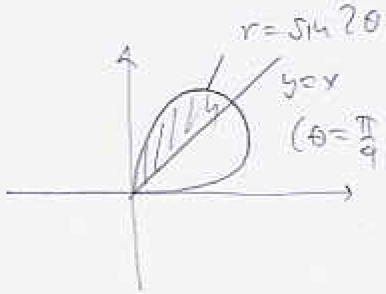
$$= \int_0^2 y^2\sqrt{1+y^3}\,dy = \begin{bmatrix} u=1+y^3 & y=2,\ u=9 \\ du=3y^2dy & y=0,\ u=1 \end{bmatrix}$$

$\frac{1}{3}du = y^2dy$

$$= \int_1^9 \frac{1}{3}\sqrt{u}\,du = \frac{1}{3}\frac{2}{3}u^{\frac{3}{2}}\Big|_1^9 = \frac{2}{9}\left(9^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$$

$$= \frac{2}{9}(27-1) = \frac{52}{9}$$

3. (10pts) Set up $\iint_D x\,dA$ in polar coordinates if D is the region inside the first-quadrant petal of the curve $r = \sin 2\theta$ that is also above the line $y = x$. Sketch the region, but do not evaluate the integral.



$$\iint_D x\,dA = \int_{\pi/4}^{\pi/2}\int_0^{\sin 2\theta} r\cos\theta \; r\,dr\,d\theta$$

$$= \int_{\pi/4}^{\pi/2}\int_0^{\sin 2\theta} r^2\cos\theta\,dr\,d\theta$$

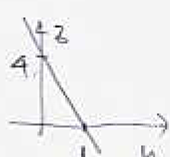
4. (12pts) Sketch the region whose volume is given by the triple integral below:

$$\int_0^1\int_{-\sqrt{2-x}}^{\sqrt{2-x}}\int_0^{4-4y} 1\,dz\,dy\,dx$$

Proj. to xy-plane

$0 \le x \le 1$

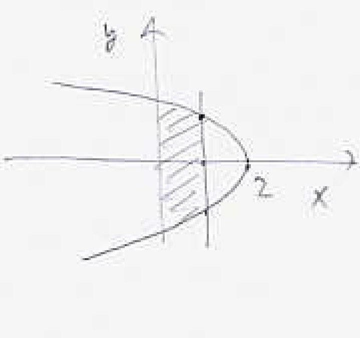
$-\sqrt{2-x} \le y \le \sqrt{2-x}$

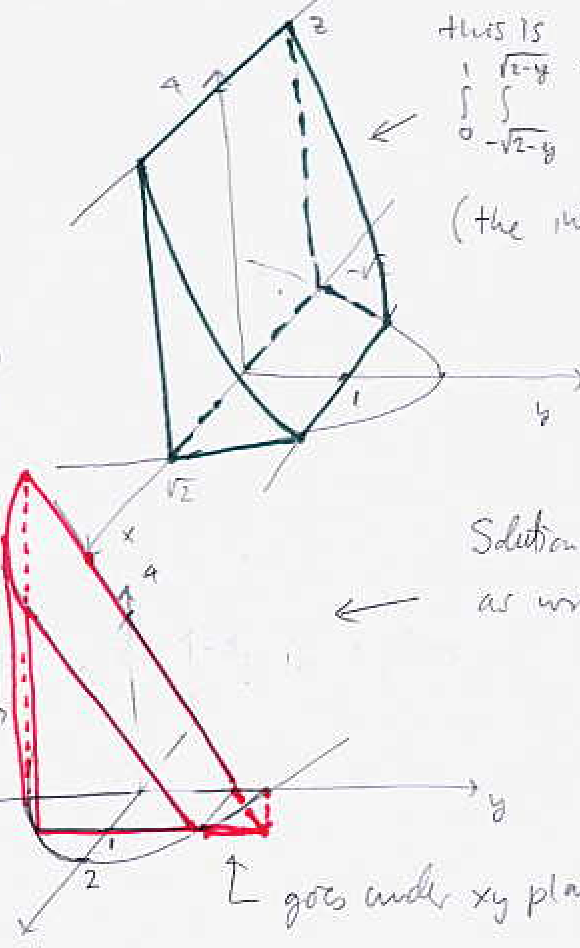


$y = \sqrt{2-x}$

$z = 4-4y$ is a plane

$x = 2-y^2$





this is $\int_0^1\int_{-\sqrt{2-y}}^{\sqrt{2-y}}\int_0^{4-4y} dz\,dx\,dy$

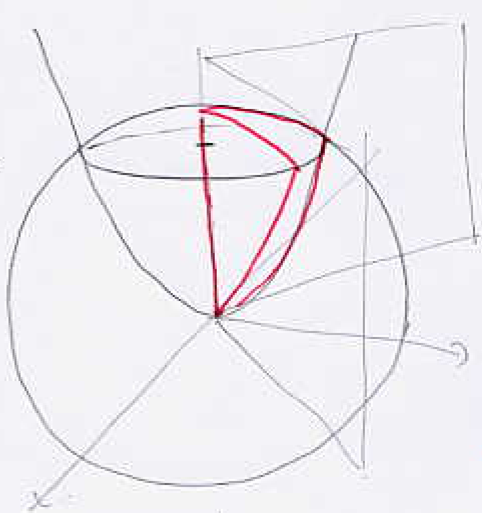
(the intended problem)

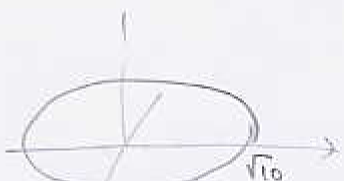
Solution to problem as written

goes under xy plane here

5. (16pts) Use cylindrical coordinates to set up $\iiint_E xyz^2\,dV$ where E is the region above the paraboloid $z = \frac{1}{2}(x^2+y^2)$, under the sphere $x^2+y^2+z^2 = 35$ and between the planes $y = \sqrt{3}x$ and $y = -\sqrt{3}x$. Sketch the region of integration. Do not evaluate the integral.

$z = 5 \rightarrow$





projects to disk of radius $\sqrt{10}$

$z = \frac{1}{2}r^2$

$r^2 + z^2 = 35$

$z^2 + 2z - 35 = 0$

$(z+7)(z-5) = 0$

$z = -7, 5$

$5 = \frac{1}{2}r^2$

$10 = r^2$

$r = \sqrt{10}$

$$\int_{\pi/3}^{2\pi/3}\int_0^{\sqrt{10}}\int_{\frac{1}{2}r^2}^{\sqrt{35-r^2}} r^3 z^2 \sin\theta\cos\theta\, dz\, dr\, d\theta$$

includes r from change of coord.

$xyz^2 = r\cos\theta\, r\sin\theta\, z^2$

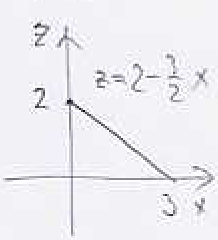
$= r^2 z^2 \sin\theta\cos\theta$

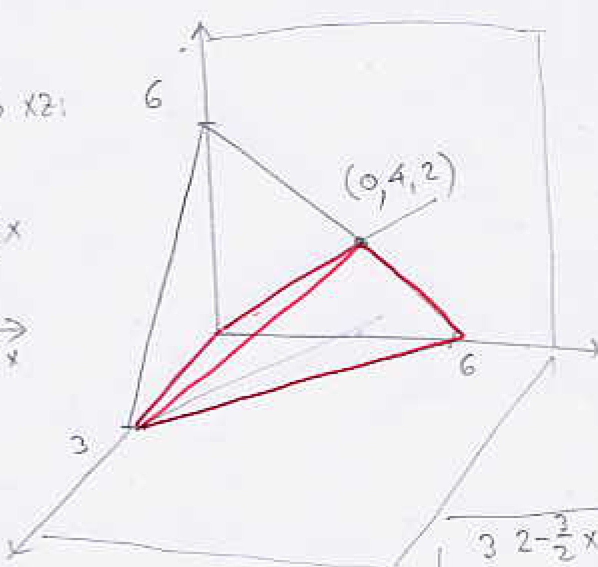
$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

6. (16pts) Sketch the region E bounded by the planes $z = 0$, $x = 0$, $2x + y + z = 6$ and $y - 2z = 0$. Then write the iterated triple integral that stands for $\iiint_E f\,dV$ that ends in $dy\,dz\,dx$.

projection to xz:





$2x+y+z=6$

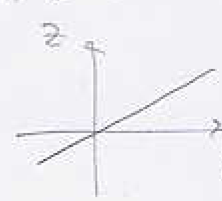
has x-int: $2x=6$, $x=3$

y-int: $y=6$

z-int: $z=6$

$y-2z=0$

$z=\frac{y}{2}$



$\begin{cases} 2x+y+z=6 \\ y-2z=0 \end{cases} \Rightarrow 2x+3z=6$

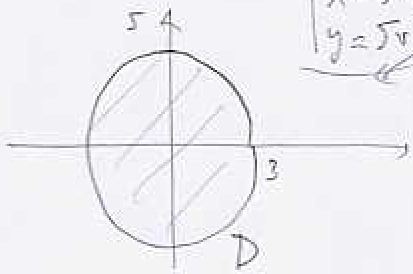
$z = 2-\frac{3}{2}x$

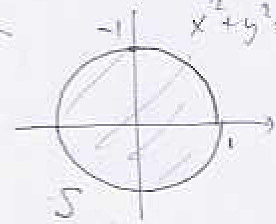
$$\boxed{\int_0^3\int_0^{2-\frac{3}{2}x}\int_{2z}^{6-2x-z} f\,dy\,dz\,dx}$$

$(x=0 \Rightarrow z=2)$

7. (14pts) Use change of variables to find the area of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

$\boxed{\begin{array}{l} x = 3u \\ y = 5v \end{array}}$





$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix} = 15$$

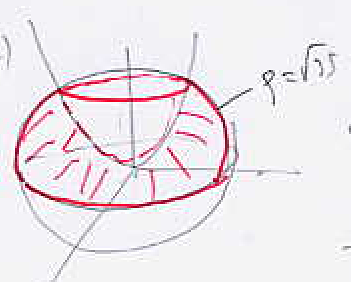
$$\iint_R 1\, dA = \iint_S 1 \cdot |15|\, du\, dv$$

$$= 15 \iint_S 1\, du\, dv = \left[\begin{array}{l} \text{polar} \\ \text{coord in } u\text{-}v \end{array} \right] =$$

$$= 15 \int_0^{2\pi} \int_0^1 r\, dr\, d\theta = 15 \cdot 2\pi \cdot \frac{1}{2} = 15\pi$$

Bonus. (10pts) Consider ther region below the paraboloid $z = \frac{1}{2}(x^2 + y^2)$, inside the sphere $x^2 + y^2 + z^2 = 35$, and above the xy-plane.
a) Set up the triple integral for the volume of this region in spherical coordinates.
b) Evaluate the integral, with final answer in exact form (not decimal!).

a)



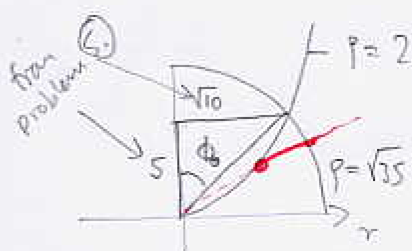
$z = \frac{1}{2} r^2$

$\rho\cos\phi = \frac{1}{2}(\rho\sin\phi)^2$

$\frac{2\cos\phi}{\sin^2\phi} = \rho$

a) $$\int_0^{2\pi} \int_{\phi_0}^{\frac{\pi}{2}} \int_{2\frac{\cos\phi}{\sin^2\phi}}^{\sqrt{35}} 1 \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

b)



$\tan\phi = \frac{\sqrt{10}}{5} = \sqrt{\frac{2}{5}}$

$\phi_0 = \arctan\sqrt{\frac{2}{5}}$

$\cot\phi_0 = \frac{5}{\sqrt{10}} = \sqrt{\frac{5}{2}}$

$$b) = 2\pi \int_{\phi_0}^{\pi/2} \sin\phi \left(\frac{\rho^3}{3} \Big|_{2\frac{\cos\phi}{\sin^2\phi}}^{\sqrt{35}} \right) d\phi \qquad \frac{\cos^3\phi}{\sin^5\phi}$$

$$= \frac{2\pi}{3} \left(\int_{\phi_0}^{\pi/2} \sqrt{35}^3 \sin\phi\, d\phi - \int_{\phi_0}^{\pi/2} 8 \sin\phi \frac{\cos^3\phi}{\sin^6\phi}\, d\phi \right)$$

$$= \frac{2\pi}{3} 35^{3/2} \cos\phi_0 - \frac{16\pi}{3} \int_{\phi_0}^{\pi/2} \csc^2\phi \cot^3\phi\, d\phi$$

$$= \left[\begin{array}{ll} u = \cot\phi & \phi = \pi/2,\ u = 0 \\ du = -\csc^2\phi\, d\phi & u = \cot\phi_0 \end{array} \right] = \boxed{\frac{325\pi}{3}}$$

$$= \frac{2\pi}{3} 35^{\frac{3}{2}} \cdot \frac{5}{\sqrt{35}} - \frac{16\pi}{3} \int_{\cot\phi_0}^{0} -u^3\, du = \frac{10\pi}{3} \cdot 35 - \frac{\cancel{16}^{\,4}\pi}{3} \cdot \frac{\cot^4\phi_0}{\cancel{4}_{\,1}} = \frac{10\pi}{3} 35 - \frac{4\pi}{3} \cdot \frac{25}{4}$$