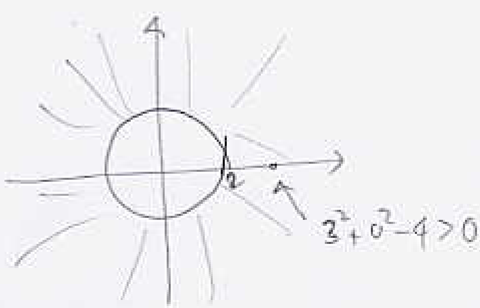
Fall '07/MAT 309/Exam 2 **Name:** - Solution *Show all your work.*

1. (13pts) Let $f(x, y) = \sqrt{x^2 + y^2 - 4}$.
a) Find the domain of f and sketch it.
b) Find the level curves for the levels $k = -1, 0, 1, 2$ and draw them.
c) The point $(1, 2)$ is on the level curve for $k = 1$. Use b) to roughly draw the vector $\nabla f(1, 2)$ emanating from the point $(1, 2)$.

a) $x^2 + y^2 - 4 \geq 0$

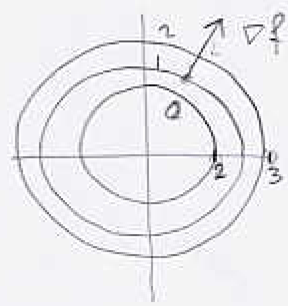
$x^2 + y^2 = 4$ is a circle



D = region outside the circle

b) $\sqrt{x^2+y^2-4} = k$ ← no curve for $k = -1$

$x^2 + y^2 = 4 + k^2$ circles of radii $2, \sqrt{5}, \sqrt{8}$ ($k = 0, 1, 2$)



∇f: normal to curve, in direction of increasing k

2. (14pts) At a cheese factory in Muscoda, WI, the intensity of the cheese smell is given by $L(x, y) = 4 + x^3 + y^3 - 3xy$.
a) Susan stands at point $(3, 1)$ and sees Joe at point $(1, 1)$. If she starts moving toward him, will the cheese smell intensify? At what rate?
b) In which direction should Susan move to experience the maximum drop in smell? At which rate will the smell intensity drop?

a) $\nabla L = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$

$\nabla L(3, 1) = \langle 27, -9 \rangle$

$\vec{a} = \langle -2, 1 \rangle$

$\vec{u} = \frac{1}{\sqrt{(-2)^2 + 1^2}} \langle -2, 1 \rangle = \frac{1}{\sqrt{5}} \langle -2, 1 \rangle$

$D_u L = \langle 27, -9 \rangle \cdot \frac{1}{\sqrt{5}} \langle -2, 1 \rangle = \frac{-54 - 9}{\sqrt{5}} = \frac{-63}{\sqrt{5}}$ intensity drops at rate $-\frac{63}{\sqrt{5}}$

Intensity drops at rate $-\frac{54}{5}$

b) In direction of $-\nabla L = \langle -27, 9 \rangle$

Then the drop is by $-|\nabla L|$

$= -\sqrt{27^2 + 9^2} = -\sqrt{810} = -28.46$

3. (9pts) A rectangular box is measured to have length $x = 30$cm, width $y = 16$cm and height $z = 90$cm, with an error in measurement at most 0.5cm in each. Use differentials to estimate the maximal error in computing the volume of the box.

$V = xyz \qquad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

$= yz\,dx + xz\,dy + xy\,dz$

$dV \approx 16\cdot 90\cdot 0.5 + 30\cdot 90\cdot 0.5 + 30\cdot 16\cdot 0.5$

$= 2310\ \text{cm}^3$ max. error

4. (16pts) Let $z = \dfrac{e^{xy}}{y}$, $x = r\cos\theta$, $y = r\sin\theta$. Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ when $r = 4$ and $\theta = \frac{\pi}{2}$.

$\frac{\partial z}{\partial x} = \frac{ye^{xy}}{y} = e^{xy}, \quad \frac{\partial z}{\partial y} = \frac{xe^{xy}y - e^{xy}}{y^2} = \frac{e^{xy}(xy-1)}{y^2} \qquad r=4,\ \theta = \frac{\pi}{2} \Rightarrow x=0,\ y=4$

$\frac{\partial z}{\partial r} = e^{xy}\cdot\cos\theta + \frac{e^{xy}}{y^2}(xy-1)\sin\theta \xrightarrow{\text{eval}} 0 + \frac{1}{16}(-1) = -\frac{1}{16}$

$\frac{\partial z}{\partial \theta} = e^{xy}\cdot(-r\sin\theta) + \frac{e^{xy}(xy-1)}{y^2}(r\cos\theta) \xrightarrow{\text{eval}} -4 + 0 = -4$

5. (15pts) The equation $xz^3 + y\ln(x+y^2z) = 7$ defines z implicitly as a function of x and y. Find the expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. $= F$

$F_x = z^3 + \frac{y}{x+y^2z}$

$F_y = \ln(x+y^2z) + \frac{2y^2z}{x+y^2z}$

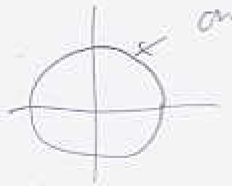
$F_z = 3xz^2 + \frac{y^3}{x+y^2z}$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{z^3 + \frac{y}{x+y^2z}}{3xz^2 + \frac{y^3}{x+y^2z}}$

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\ln(x+y^2z) + \frac{2y^2z}{x+y^2z}}{3xz^2 + \frac{y^3}{x+y^2z}}$

6. (15pts) A circle of radius 3 is parametrized by $x = 3\cos t$, $y = 3\sin t$.
a) Find the circumference of the circle using the formula for length of parametric curves.
b) Reparametrize the curve with respect to arc length, measured from the point for which $t = 0$.

a)



one circle for $0 < t < 2\pi$

$x' = -3\sin t$

$y' = 3\cos t$

$$\int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{9}\,dt = 6\pi$$

b)

$$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t 3\,du = 3t \qquad t = \frac{s}{3}$$

$x = 3\cos\frac{s}{3}$

$y = 3\sin\frac{s}{3}$

7. (18pts) Find and classify the local extremes for the function $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.

$f_x = 6x^2 + y^2 + 10x$

$f_y = 2xy + 2y$

$$D = \begin{vmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{vmatrix}$$

$2xy + 2y = 0$

$2y(x+1) = 0$

$y = 0$ or $x = -1$

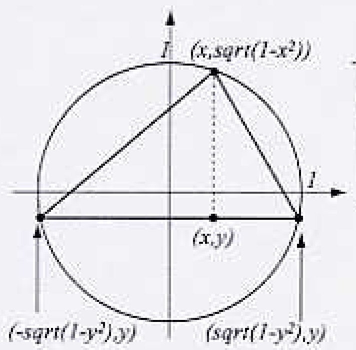
$\Rightarrow 6x^2 + 10x = 0$ $\qquad y^2 - 4 = 0$

$2x(3x+5) = 0$ $\qquad y = \pm 2$

$x = 0$ or $x = -\frac{5}{3}$

crit pts	$D(x,y)$	
$(0,0)$	20	local min, since $f_{xx}(0,0) > 0$
$(-\frac{5}{3}, 0)$	$-10 \cdot (-\frac{4}{3}) = \frac{40}{3}$	local max since $f_{xx}(-\frac{5}{3}, 0) < 0$
$(-1, 2)$	-16	saddle pts.
$(-1, -2)$	-16	saddle pts.

Bonus. (12pts) Among all triangles inscribed in the unit circle, find the one with the largest area. Tips:
1) By rotating, the triangle can be positioned so that one of its heights is vertical and the vertex on the height is on the upper half-circle, like in the picture.
2) Write the area $A(x, y)$ of the triangle in terms of x and y from the picture. Your job is to maximize $A(x, y)$ over a certain closed region in $\mathbf{R}^2$. What is the region? (Note: the triangle is completely determined by the point (x, y) where the height meets the base.)
3) What are the critical points of $A(x, y)$? (Squaring A won't help much. Find A_x and A_y directly.) Then investigate values on the boundary of the region.

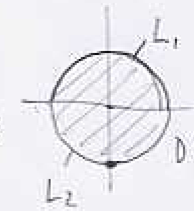


$\sqrt{1-x^2}-y$

$2\sqrt{1-y^2}$

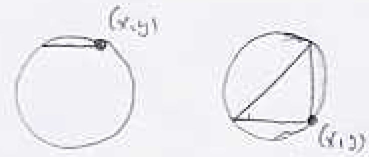
$A = \frac{1}{2} b \cdot h = \frac{1}{2} 2\sqrt{1-y^2}\left(\sqrt{1-x^2}-y\right)$

$$A(x,y) = \sqrt{1-y^2}\sqrt{1-x^2} - y\sqrt{1-y^2}$$



Constraint: (x,y) has to be in unit circle

Ex:



area = 0 | biggest triangle

$A_x = \sqrt{1-y^2}\left(\frac{-2x}{2\sqrt{1-x^2}}\right) = -\frac{x\sqrt{1-y^2}}{\sqrt{1-x^2}}$

$A_y = -\frac{2y}{2\sqrt{1-y^2}}\sqrt{1-x^2} - \left(\sqrt{1-y^2} - \frac{2y^2}{2\sqrt{1-y^2}}\right)$

$A_x = 0 \Rightarrow x = 0$ or $y^2 = 1$, $y = \pm 1$ not in interior

$(A_y = 0) \Rightarrow -\frac{y}{\sqrt{1-y^2}} - \sqrt{1-y^2} + \frac{y^2}{\sqrt{1-y^2}} = 0 \quad | \cdot \sqrt{1-y^2}$

$-y - (1-y^2) + y^2 = 0$

$2y^2 - y - 1 = 0$

$y = \frac{1 \pm \sqrt{1+8}}{4}$

$= 1, -\frac{1}{2}$

↑ not in interior

$A(0, -\frac{1}{2})$

$= \sqrt{\frac{3}{4}} - \left(-\frac{1}{2}\right)\sqrt{\frac{3}{4}}$

$= \frac{3}{2}\sqrt{\frac{3}{4}} = \frac{3\sqrt{3}}{4}$

On boundary:

L_1: $y = \sqrt{1-x^2} \quad x \in [-1, 1]$

$A(x,y) = 0$

L_2: $y = -\sqrt{1-x^2}$

$\Rightarrow \sqrt{1-y^2} = \sqrt{x^2} = |x|$

$A(x,y) = |x|\sqrt{1-x^2} + \sqrt{1-y^2}\,|x|$

$= 2|x|\sqrt{1-x^2}$

$g(x) = \left(\frac{1}{2}A(x,y)\right)^2 = x^2(1-x^2) = x^2 - x^4$

$g'(x) = 2x - 4x^3$

$2x - 4x^3 = 0$

$x(1-2x^2) = 0$

$x = 0$ or $x = \pm\sqrt{\frac{1}{2}}$

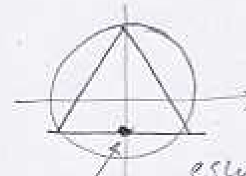
$y = -1$ | $y = -\sqrt{\frac{1}{2}}$

$A = 0$ | $A = 2 \cdot \sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}} = 1$

Check candidates for maximum

(x,y)	$A(x,y)$	
$(0, -\frac{1}{2})$	$\frac{3\sqrt{3}}{4}$	max
$(0, -1)$	0	
$(\pm\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$	1	

Biggest area belongs to triangle



equilateral triangle