Fall '07/MAT 309/Exam 1 **Name:** Solution *Show all your work.*

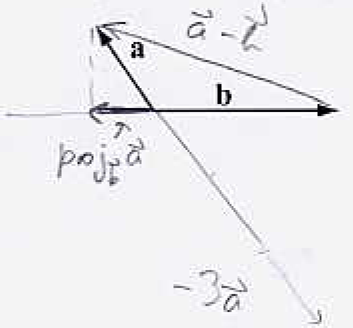
1. (6pts) Find $\langle 3, 2+t, 5-2t\rangle \times \langle 2, 1, t+1\rangle =$

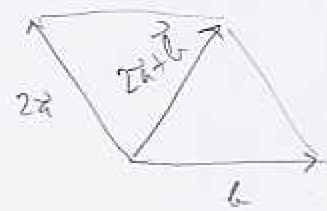
$$\begin{vmatrix} \vec i & \vec j & \vec k \\ 3 & 2+t & 5-2t \\ 2 & 1 & t+1 \end{vmatrix} = \big((2+t)(t+1)-(5-2t)\big)\vec i - \big(3(t+1)-2(5-2t)\big)\vec j + \big(3-2(2+t)\big)\vec k$$

$$= (t^2+3t+2-5+2t)\vec i - (3t+3-10+4t)\vec j + (3-4-2t)\vec k$$

$$= (t^2+5t-3)\vec i + (-7t+7)\vec j + (-1-2t)\vec k$$

2. (17pts) Let **a** and **b** be vectors sketched below.
a) draw the vectors $-3\mathbf{a}$, $\mathbf{a}-\mathbf{b}$ and $2\mathbf{a}+\mathbf{b}$.
b) draw the vector $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$.
c) If $\mathbf{a} = -2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = 7\mathbf{i}$, find the coordinates of $\mathrm{proj}_{\mathbf{b}}\,\mathbf{a}$.





c) $\mathrm{proj}_{\vec b}\vec a = \dfrac{\vec b\cdot\vec a}{|\vec b|^2}\vec b = \dfrac{-14}{49}\,7\vec i = -2\vec i$

3. (6pts) Let **a**, **b** and **c** be vectors, and u, v scalars. Are the following expressions defined? For those that are not, explain what is wrong.

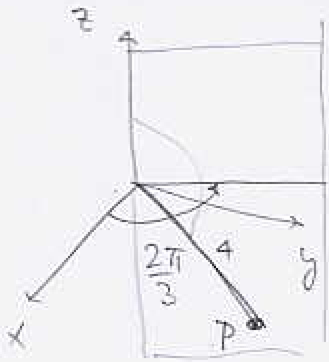
$\mathbf{a} \times (\mathbf{b}\cdot\mathbf{c})$

vector × scalar
is not defined

$(\mathbf{b}\cdot\mathbf{c})\mathbf{a}\times\mathbf{c}$

scalar (vector) × vector
is defined

4. (10pts) The spherical coordinates of a point are $\left(4, \frac{2\pi}{3}, \frac{3\pi}{4}\right)$. Sketch the point and find its cylindrical coordinates (exact numbers, not decimal).



$z = \rho\cos\phi = 4\cos\frac{3\pi}{4} = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

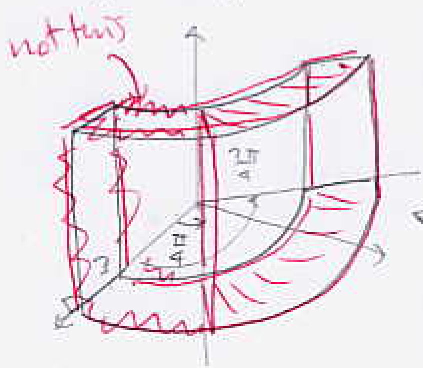
$r^2 + z^2 = \rho^2$ $\qquad r^2 = 8$

$r^2 + (-2\sqrt{2})^2 = 4^2$ $\qquad r = \sqrt{8} = 2\sqrt{2}$

$r^2 + 8 = 16$ $\qquad$ cyl. coord: $\left(2\sqrt{2}, \frac{2\pi}{3}, -2\sqrt{2}\right)$

5. (9pts) In cylindrical coordinates, draw the solid described by:

$3 \le r \le 5,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4},\ 0 \le z \le 4$



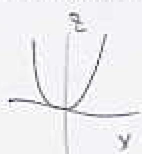
thick cylindrical slab

6. (13pts) This problem is about the surface $y = \frac{x^2}{9} + \frac{z^2}{16}$.

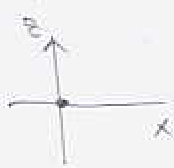
a) Sketch and identify the intersections of this surface with the coordinate planes.

b) Sketch the surface in 3D, with coordinate system visible.

a) $x=0$ $\quad y = \frac{z^2}{16}$

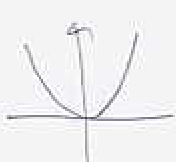


$y=0$ $\quad \frac{x^2}{9} + \frac{z^2}{16} = 0$

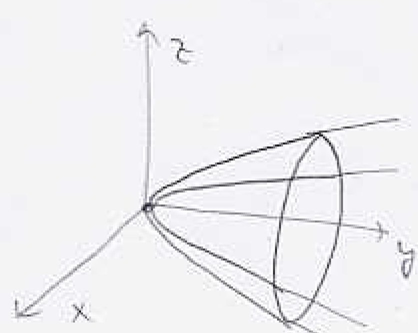


one point

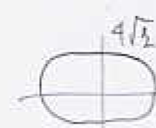
$z=0$ $\quad y = \frac{x^2}{9}$



parabola



paraboloid



(intersection with $y=k$: $\frac{x^2}{9} + \frac{z^2}{16} = k$ $\quad \frac{x^2}{9k} + \frac{z^2}{16k} = 1$ is an ellipse)

7. (9pts) The parametric equations of a curve are $x = 3\cos t$, $y = t^2$, $z = 3\sin t$, $0 \le t \le 6\pi$. Sketch this curve.

$x = 3\cos t$

$z = 3\sin t$

rotation in xz plane (3 loops)
while advancing
in y-direction

further and further apart

8. (17pts) A jet-powered eggplant travels along the path $x = t^2 + 2t$, $y = \dfrac{3}{t+1}$, $z = t^2 e^t$. At the point $(8, 1, 4e^2)$ it experiences engine failure, so from this point on, it continues along the tangent line to this curve and splatters on the xz-plane.
a) Find the parametric equations of the line tangent to the curve at $(8, 1, 4e^2)$.
b) At which point does the tangent line intersect the xz-plane? (This is where the eggplant splatters if gravitation is ignored.)

a) $t^2 + 2t = 8 \qquad 2^2 + 4 = 8$

$\dfrac{3}{t+1} = 1 \qquad t+1 = 3,\ t = 2$

$t^2 e^t = 4e^2 \qquad 2^2 e^2 = 4e^2.$

$t = 2$

$\vec{r}\,'(t) = \left\langle 2t+2,\ -\dfrac{3}{(t+1)^2},\ 2te^t + t^2 e^t \right\rangle$

$\vec{r}\,'(2) = \left\langle 6,\ -\dfrac{3}{9},\ 4e^2 + 4e^2 \right\rangle$

$= \left\langle 6,\ -\dfrac{1}{3},\ 8e^2 \right\rangle$

param equations are

$x = 8 + 6t$

$y = 1 + \frac{1}{3}t$

$z = 4e^2 + 8e^2 t$

b) Intersection with xz-plane:

$y = 0$

$1 - \frac{1}{3}t = 0$

$t = 3$

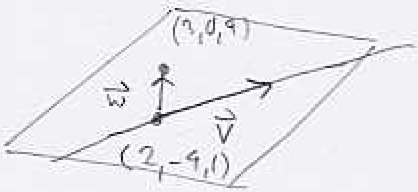
pt. is $(26, 0, 28e^2)$

9. (13pts) Find the equation of the plane that contains the line $\dfrac{x-2}{1} = \dfrac{y+4}{-3} = \dfrac{z-1}{2}$ and the point $(3, 0, 4)$.

$x = 2 + t$

$y = -4 - 3t$

$z = 1 + 2t$



$$\vec{n} = \vec{w} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 3 \\ 1 & -3 & 2 \end{vmatrix} = (8+9)\vec{i} - (2-3)\vec{j} + (-3-4)\vec{k}$$

$$= 17\vec{i} + \vec{j} - 7\vec{k}$$

need $\vec{w} = \langle 1, 4, 3 \rangle$

$\vec{v}$ = direction of line

$= \langle 1, -3, 2 \rangle$

Equation of plane:

$17(x-3) + (y-0) - 7(z-4) = 0$

$17x + y - 7z = 23$

Bonus. (10pts) This problem is about the vector $\mathbf{d} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

a) Explain why $\mathbf{d}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$.

b) Since $\mathbf{d}$ lies in the plane spanned by $\mathbf{b}$ and $\mathbf{c}$, it can be written as $\mathbf{d} = u\mathbf{b} + v\mathbf{c}$, for some scalars u and v. Find a relationship between u and v by dotting the equation $\mathbf{d} = u\mathbf{b} + v\mathbf{c}$ by $\mathbf{a}$.

a) $\vec{b} \times \vec{c}$ is perpendicular to both $\vec{b}$ and $\vec{c}$ so is parallel to normal vector of plane spanned by $\vec{b}$ and $\vec{c}$.

$\vec{a} \times (\vec{b} \times \vec{c})$ is perpendicular to both $\vec{a}$ and $\vec{b} \times \vec{c}$.

Perp. to $\vec{b} \times \vec{c}$ means that it is in plane spanned by $\vec{b}$ and $\vec{c}$.

b) $\vec{d} = \vec{a} \times (\vec{b} \times \vec{c}) = u\vec{b} + v\vec{c} \quad | \cdot \vec{a}$

$$\underbrace{(\vec{a} \times (\vec{b} \times \vec{c}))}_{} \cdot \vec{a} = u(\vec{b} \cdot \vec{a}) + v(\vec{c} \cdot \vec{a})$$

$$0 = u(\vec{b} \cdot \vec{a}) + v(\vec{c} \cdot \vec{a})$$

since $\vec{a}$ and $\vec{a} \times (\vec{b} \times \vec{c})$ are perpendicular

$$v = -\frac{\vec{b} \cdot \vec{a}}{\vec{c} \cdot \vec{a}} u$$