Spring '05/MAT 250/Final Exam **Name:** Solution *Show all your work.*

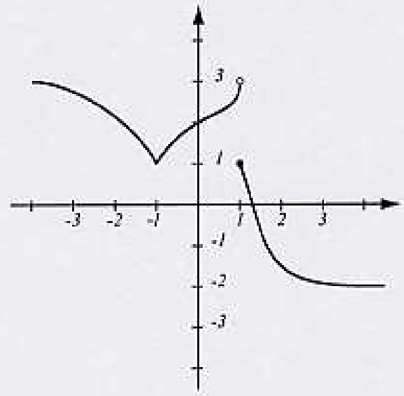
1. (5pts) Use the graph of the function f at right to answer:

a) At which points x is f not continuous? $x=1$

b) At which points x is f not differentiable? $x=-1, 1$

c) $\lim\limits_{x\to 1-} f(x) = 3$

d) $\lim\limits_{x\to\infty} f(x) = -2$



Differentiate and simplify where appropriate:

2. (4pts) $\frac{d}{dx}(x^3+4x^2+1)e^{2x} = (3x^2+8x)e^{2x} + (x^3+4x^2+1)e^{2x}\cdot 2$

$$= (2x^3+11x^2+8x+2)\,e^{x}$$

3. (3pts) $\frac{d}{dx}\sin(\ln x) = \cos(\ln x)\cdot\frac{1}{x} = \frac{\cos(\ln x)}{x}$

4. (3pts) The edge of a cube was found to be 30cm with a possible error in measurement of 0.1cm. Use differentials to estimate the maximum possible error in computing the volume of the cube.

$V = x^3$

$dV = 3x^2\,dx$

$\Delta V \approx dV = 3\cdot 30^2\cdot 0.1$

$= 2700\cdot 0.1$

$= 270 \text{ cm}^3$

5. (4pts) Find the limit algebraically. Do not use L'Hospital's rule.

a) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x\to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$

6. (4pts) Use L'Hospital's rule to find the limit:

$\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}} \overset{LH}{=} \lim_{x\to 0+} \frac{\cos x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0+} 2\sqrt{x}\cos x = 2\cdot 0\cdot 1 = 0$

$\frac{0}{0}$

7. (4pts) Evaluate.

$\int_0^8 (e^{-x} + \sqrt[3]{x})\,dx = \left(-e^{-x} + \frac{3}{4}x^{\frac{4}{3}}\right)\Big|_0^8 = -(e^{-8} - e^{0}) + \frac{3}{4}\left(8^{4/3} - 0\right)$

$x^{\frac{1}{3}}$

$= 1 - \frac{1}{e^8} + \frac{3}{4}\cdot 16 = 13 - \frac{1}{e^8} = 12.9997$

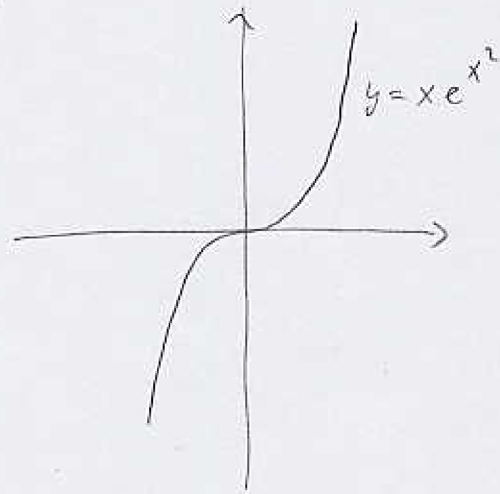
8. (5pts) Use substitution to find:

$\int \frac{\sec^2 x}{\tan x}\,dx = \begin{bmatrix} u = \tan x \\ du = \sec^2 x\,dx \end{bmatrix} = \int \frac{1}{u}\,du = \ln|u| = \ln|\tan x| + C$

9. (8pts) Let $f(x) = xe^{x^2}$.
a) Find the intervals of increase/decrease and where f has a local maximum and minimum.
b) Find the intervals where f is concave up or down.
c) Use your calculator and the results of a) and b) to accurately sketch the graph of f.

$$f'(x) = 1 \cdot e^{x^2} + x \cdot e^{x^2} \cdot 2x$$
$$= (1+2x^2)e^{x^2}$$
$$f''(x) = 4x\, e^{x^2} + (1+2x^2)e^{x^2} \cdot 2x$$
$$= (6x + 4x^3)\, e^{x^2}$$
$$= x(6+4x^2)e^{x^2}$$

c)



a) $f'(x) = 0$

$$\underbrace{(1+2x^2)}_{\text{always } >0}\underbrace{e^{x^2}}_{>0} = 0 \quad \text{no solution}$$

Hence, $f'(x) > 0$ for all x

f is always increasing

b) $f''(x) = 0$

$$x\underbrace{(6+4x^2)}_{\substack{>0 \\ \text{always}}}\underbrace{e^{x^2}}_{>0} = 0$$

$x = 0$

		0	
f''	$-$	0	$+$
f	conc dn	inflection	conc up

10. (6pts) Use implicit differentiation to find y'.

$x\cos y = 4 + x^2y^2 \quad \Big| \frac{d}{dx}$

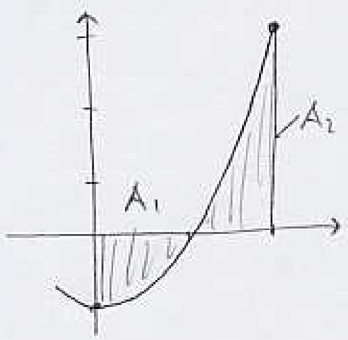
$\cos y - x\sin y\, y' = 2xy^2 + x^2 2yy'$

$-x\sin y\, y' - 2x^2 y y' = 2xy^2 - \cos y$

$y'(-x\sin y - 2x^2 y) = 2xy^2 - \cos y$

$y' = \dfrac{\cos y - 2xy^2}{x\sin y + 2x^2 y}$

11. (4pts) Use a graph to determine whether $\int_0^2 (x^2-1)\,dx$ is positive or negative. Then evaluate the integral and verify your answer.



$\int_0^2 (x^2-1)\,dx = A_2 - A_1 > 0$ because A_2 appears to be larger.

Indeed: $\int_0^2 x^2-1\,dx = \left(\frac{x^3}{3} - x\right)\Big|_0^2 = \frac{8}{3} - 2 - 0 = \frac{2}{3}$

12. (3pts) The velocity of a jet-ski is given by $v(t) = 3 + t$ meters per second. By how much did it change position from time $t = 2$ to $t = 4$?

$\Delta S = \int_2^4 (3+t)\,dt = \left(3t + \frac{t^2}{2}\right)\Big|_2^4 = 3(4-2) + \frac{1}{2}(4^2 - 2^2)$

$= 6 + \frac{1}{2}\cdot 12 = 12\text{ m}$

13. (6pts) Use the closed interval method to find the absolute minimum and maximum values for the function $f(x) = 2x^3 + 3x^2 - 36x + 17$ on the interval $[0, 4]$.

$f'(x) = 6x^2 + 6x - 36$

$6x^2 + 6x - 36 = 0$

$x^2 + x - 6 = 0$

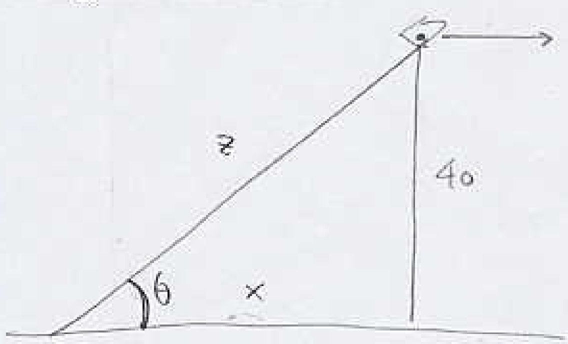
$(x-2)(x+3) = 0$

$x = 2, -3$

-3 not in $[0,4]$

x	$f(x)$	
2	$16 + 12 - 72 + 17 = -27$	abs. min
0	17	
4	$128 + 48 - 144 + 17 = 49$	abs max.

14. (7pts) A kite is moving horizontally at altitude 40 meters and speed 2 meters per second. At what rate is the angle between the string and the horizontal decreasing when 80m of string have been let out?



Know: $\frac{dx}{dt} = 2$

Need $\frac{d\theta}{dt}$ when $z = 80$

$\tan\theta = \frac{40}{x}$ $\Big| \frac{d}{dt}$

$\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{40}{x^2} \cdot \frac{dx}{dt}$

$\frac{x}{z} = \cos\theta$

$\frac{x}{\cos\theta} = z$

$z = x\sec\theta$

$\frac{d\theta}{dt} = -\frac{40}{\underbrace{(x\sec\theta)}_{z}^2}\frac{dx}{dt} = -\frac{40}{z^2}\frac{dx}{dt}$

when $z = 80$, $\frac{d\theta}{dt} = -\frac{40}{80^2} \cdot 2 = -\frac{1}{80}$ rad/s

15. (4pts) Use the Intermediate Value Theorem to show that the equation $x - \cos x = 0$ has a solution in the interval $[0, \pi/2]$.

Let $f(x) = x - \cos x$

f is continuous on all of $\mathbb{R}$,

$f(0) = 0 - \cos 0 = -1$

$f(\frac{\pi}{2}) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2}$

Since $-1 < 0 < \frac{\pi}{2}$, by IVT

there is a number c in $[0, \frac{\pi}{2}]$

so that $f(c) = 0$.

Bonus 1. (4pts) Find a formula for the n-th derivative of $f(x) = \dfrac{1}{x^4}$.

$y = x^{-4}$

$y' = -4x^{-5}$

$y'' = (-4)(-5)x^{-6}$

$y''' = (-4)(-5)(-6)x^{-7}$

$y'''' = (-4)(-5)(-6)(-7)x^{-8}$

$y^{(n)} = (-4)(-5)\ldots(-(n+3))\,x^{-(n+4)}$

$= \dfrac{(-1)^n\, 4 \cdot 5 \cdots (n+3)}{x^{n+4}}$

Bonus 2. (4pts) A toll-road ticket shows a motorist entering a 160-mile long highway at 1:00PM and exiting at 3:00PM. If the speed limit on the road is 65mph, explain how a policeman armed with the Mean Value Theorem can prove that the motorist was speeding.

Let $f(t)$ = position at time t.

f is a continuous and differentiable function, $f(1) = 0$

$f(3) = 160$

By MVT, there exists a number c in $[1, 3]$ so that

$f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{160 - 0}{2} = 80$

Thus, at some pt. during trip velocity was 80 mph.