Fall '04/MAT 250/Final Exam Name: Solution *Show all your work.*

Differentiate and simplify where appropriate:

1. (5pts) $\frac{d}{dx}\frac{x+3}{x^2-4} = \frac{1\cdot(x^2-4)-(x+3)\cdot 2x}{(x^2-4)^2} = \frac{x^2-4-(2x^2+6x)}{(x^2-4)^2} = \frac{-x^2-6x-4}{(x^2-4)^2}$

2. (4pts) $\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\cos x = \cot x$

3. (4pts) Find the limit algebraically. Do not use L'Hospital's rule.

a) $\lim_{x\to 4}\frac{x^2-2x-8}{x-4} = \lim_{x\to 4}\frac{(x+2)\cancel{(x-4)}}{\cancel{x-4}} = 4+2 = 6$

4. (4pts) Use L'Hospital's rule to find the limit:

$\lim_{x\to 0}\frac{e^x-1-x}{x^2} \underset{\text{L'H}}{=} \lim_{x\to 0}\frac{e^x-1}{2x} \underset{\text{L'H}}{=} \lim_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$

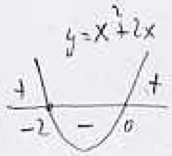
$\frac{e^0-1-0}{0} = \frac{0}{0}$ $\frac{0}{0}$

5. (8pts) Let $f(x) = x^2e^x$.
a) Find the intervals of increase/decrease and where f has a local maximum and minimum.
b) Find the intervals where f is concave up or down.
c) Use your calculator and the results of a) and b) to accurately sketch the graph of f.

a) $f'(x) = 2xe^x + x^2e^x$
$= e^x(2x + x^2)$

$e^x(2x+x^2) = 0 \quad e^x \neq 0$
$x(2+x) = 0$
$x = 0$ or -2

$y = x^2+2x$



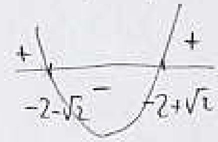
		-2		0	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$	loc. max	$\searrow$	loc. min	$\nearrow$

b) $f''(x) = e^x(2x+x^2) + e^x(2+2x)$
$= e^x(x^2+4x+2)$

$x^2+4x+2 = 0$

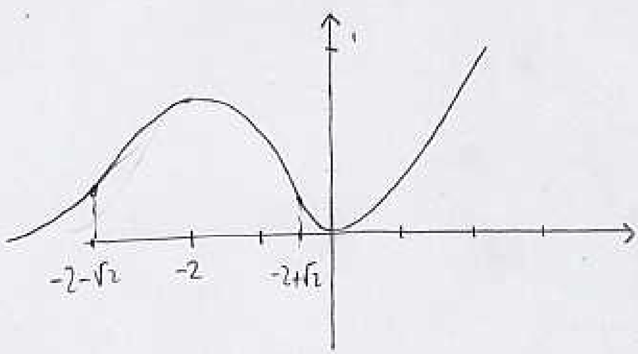
$x = \dfrac{-4 \pm \sqrt{16-4\cdot 2}}{2} = \dfrac{-4 \pm \sqrt{8}}{2} = \dfrac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$

$y = x^2+4x+2$



		$-2-\sqrt{2}$		$-2+\sqrt{2}$	
f''	$+$	0	$-$	0	$+$
f	CU		CD		CU

inflection points



6. (3pts) Find $D^{65} \sin 3x$.

$65 = 64 + 1$
16 cycles

$D^{65} \sin 3x = 3^{65} \cos 3x$

$y = \sin 3x$
$y' = 3\cos 3x$

7. (8pts) Find the following definite and indefinite integrals. Use substitution in the second one.

a) $\displaystyle\int_1^3 \frac{4x^2-1}{x}\,dx = \int_1^3 \frac{4x^2}{x} - \frac{1}{x}\,dx = \int_1^3 4x - \frac{1}{x}\,dx = \left(4\,\frac{x^2}{2} - \ln|x|\right)\Big|_1^3$

$= 2(3^2-1^2) - (\ln 3 - \ln 1) = 16 - \ln 3$

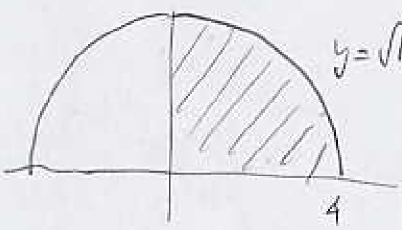
b) $\displaystyle\int \frac{\sin x}{\cos^2 x}\,dx = \begin{bmatrix} u = \cos x \\ du = -\sin x\,dx \end{bmatrix} = \int \frac{1}{u^2}(-du) = \int -u^{-2}\,du = -\frac{u^{-1}}{-1} = \frac{1}{u}$

$-du = \sin x\,dx$

$= \dfrac{1}{\cos x} + C$

8. (4pts) Interpret the following integral as area to help you find it. Draw a picture.

$\displaystyle\int_0^4 \sqrt{16-x^2}\,dx =$ shaded area $= \frac{1}{4}\,4^2\cdot\pi = 4\pi$

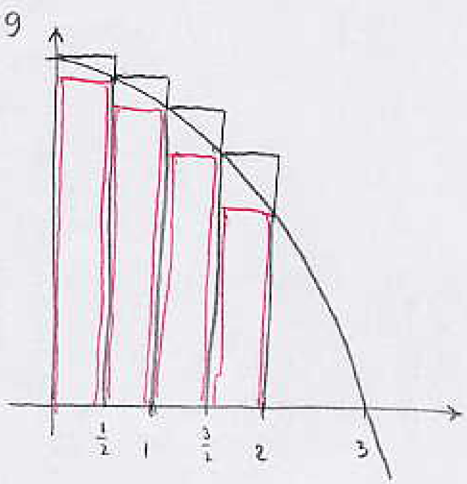


$y = \sqrt{16-x^2}$ upper half of circle $x^2+y^2=16$

(radius = 4

9. (7pts) Use four rectangles to estimate the area under the curve $y = 9 - x^2$ from $x = 0$ to $x = 2$. Choose sample points in two ways (draw a picture big and beautiful) so that you
a) Overestimate the area.
b) Underestimate the area.

62
68



a) L_4 overestimates (black)

$$L_4 = \underbrace{\left(9 + \frac{35}{4} + 8 + \frac{27}{4}\right)}_{f \text{ evaluated at } 0, \frac{1}{2}, 1, \frac{3}{2}} = \frac{130}{8}$$

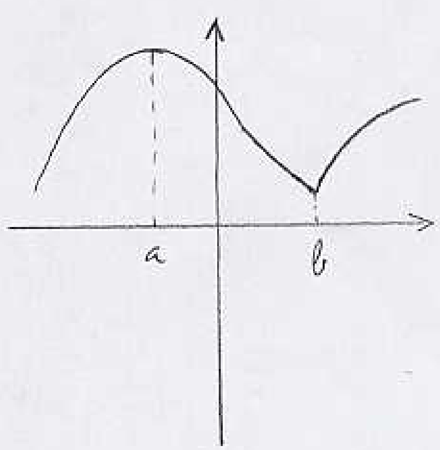
$$= \frac{65}{4} = 16.25$$

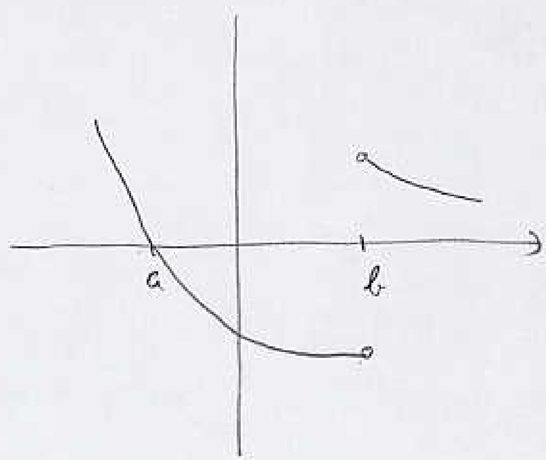
b) R_4 underestimates (red)

$$R_4 = \underbrace{\left(\frac{35}{4} + 8 + \frac{27}{4} + 5\right)}_{f \text{ evaluated at } \frac{1}{2}, 1, \frac{3}{2}, 2} \frac{1}{2} = \frac{114}{8}$$

$$= \frac{57}{4} = 14.25$$

10. (4pts) The graph of f is given. Sketch the graph of f'.





11. (7pts) Consider the equation $e^x - x - 2 = 0$.
a) Use the Intermediate Value Theorem to show that this equation has a solution in the interval [0,2].
b) Use your calculator to find an interval of length 0.01 that contains this root. Explain why the IVT will guarantee there is root in the interval that you found.

a) Let $f(x) = e^x - x - 2$

$f(0) = -1 < 0$

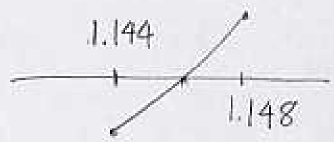
$f(2) = e^2 - 4 = 3.39 > 0$

Since $-2 < 0 < e^2 - 4$, by IVT

there is a number c in $(0,2)$

so that $f(c) = 0$.

b) Zooming in successively we find:



x	$f(x)$
1.144	-0.005
1.148	0.005

Since $-0.005 < 0 < 0.005$, by IVT

there is a number c in $(1.144, 1.148)$

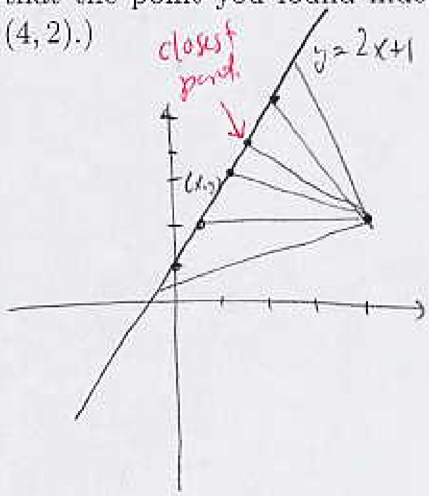
so that $f(c) = 0$

12. (4pts) Oil is leaking from a tank at rate $3 + 2t$ liters per minute through a hole that is increasing in size. How much oil leaks out from $t = 4$ minutes to $t = 7$ minutes?

$$\text{total leaked} = \int_4^7 \text{rate of leaking}\, dt = \int_4^7 3 + 2t\, dt = \left(3t + t^2\right)\Big|_4^7$$

$$= 3(7-4) + (49-16) = 9 + 33 = 42 \text{ liters}$$

13. (8pts) Find the point on the line $y = 2x + 1$ that is closest to the point $(4, 2)$. Verify that the point you found indeed is the closest. (Hint: minimize the square of distance to $(4, 2)$.)



$d((x,y),(4,2)) = \sqrt{(x-4)^2 + (y-2)^2}$

$f(x) = d^2 = (x-4)^2 + (2x+1-2)^2 = (x-4)^2 + (2x-1)^2$

Job: minimize f on $(-\infty, \infty)$

$y = 10x - 12$

$f'(x) = 2(x-4) + 2(2x-1)\cdot 2$

$= 10x - 12$

$\frac{6}{5}$

f' $-$ 0 $+$

f $\searrow$ loc. min $\nearrow$

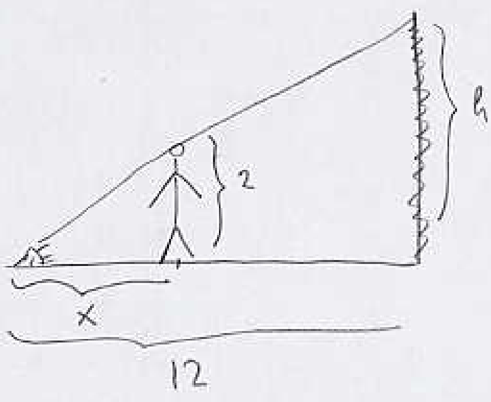
$10x - 12 = 0$

$x = \frac{12}{10} = \frac{6}{5}$

$y = \frac{12}{5} + 1 = \frac{17}{5}$

Since f has a local min at $\frac{6}{5}$, and it is the only one on an open interval, it is an absolute minimum.

Bonus. (7pts) A spotlight on the ground shines on a wall 12m away. If a man 2m tall walks from the spotlight toward the building at speed of 1.6m/s, how fast is the length of his shadow on the building decreasing when he is 4m from the building?



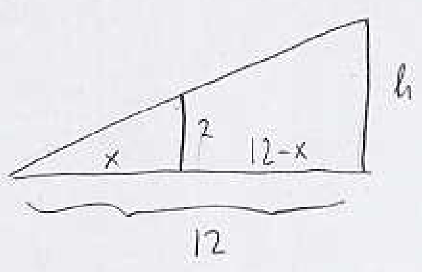
Know: $\frac{dx}{dt} = 1.6$ m/s

Need: $\frac{dh}{dt}$

Similar triangles give: $\frac{x}{2} = \frac{12}{h}$

$h = \frac{24}{x}$ $\Big| \frac{d}{dt}$

$\frac{dh}{dt} = -\frac{24}{x^2} \cdot \frac{dx}{dt}$



Put in $x = 8$ (when man is 4m from building)

$\frac{dh}{dt} = -\frac{24}{8^2} \cdot 1.6 = -\frac{3}{8} \cdot 1.6$ m/s

$= -3 \cdot 0.2 = -0.6$ m/s