Fall '04/MAT 250/Exam 3 **Name:** Solution *Show all your work.*

1. (9pts) Use L'Hospital's rule to find the limits:

a) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = \frac{2}{\infty} = 0$

$\frac{\ln\infty}{\sqrt{\infty}} = \frac{\infty}{\infty}$

b) $\underbrace{\lim_{x\to 0}(1+x)^{-\frac{1}{x}}}_{y} = e^{-1} = \frac{1}{e}$

$1^{-\infty}$

$\ln y = -\frac{1}{x}\ln(1+x)$

$\lim_{x\to 0} -\frac{1}{x}\ln(1+x) = \lim_{x\to 0} -\frac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{x\to 0} -\frac{\frac{1}{1+x}}{1} = -1$

$\frac{\ln 1}{0} = \frac{0}{0}$

2. (6pts) The function $f(x) = \sqrt{x}$ is given.
a) Find the linearization of this function around the point $a = 25$.
b) Use the linearization to estimate $\sqrt{26}$. How far is your estimate from $\sqrt{26}$?

$L(x) = f(25) + f'(25)(x-25)$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$

$= 5 + \frac{1}{10}(x-25)$

$\sqrt{26} \approx L(26) = 5 + \frac{1}{10}(26-25) = 5.1$

$\sqrt{26} = 5.099019514$ quite close

3. (10pts) Let $f(x) = x^3 + 3x^2 - 24x + 2$.

a) Find the intervals of increase/decrease and where f has a local maximum and minimum.

b) Find the intervals where f is concave up or down.

c) Use your calculator and the results of a) and b) to accurately sketch the graph of f.

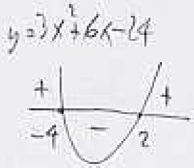
a) $f'(x) = 3x^2 + 6x - 24$

$3x^2 + 6x - 24 = 0 \quad | \div 3$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$x = -4, 2$

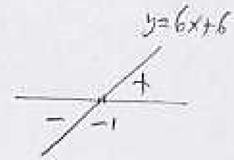


		-4		2	
f'	+	0	−	0	+
f	↗	loc. max	↘	loc. min	↗

b) $f''(x) = 6x + 6$

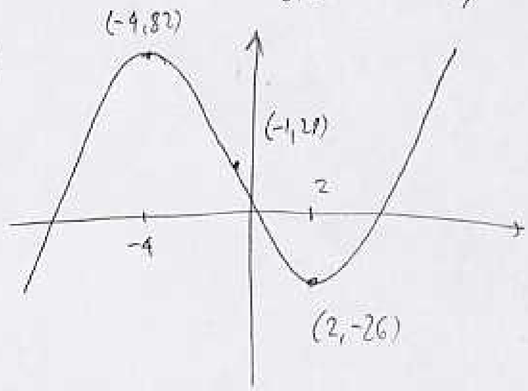
$6x + 6 = 0$

$x = -1$



		-1	
$6x+6$	−		+
f	conc. dn		conc. up

c)



4. (5pts) Suppose that for a continuous and differentiable function f we have $-1 \le f'(x) \le 2$ for all x in $[3, 5]$ and $f(3) = 7$. Use the Mean Value Theorem to show that $5 \le f(5) \le 11$.

By MVT there exists a c in $(3,5)$ so that

$$f'(c) = \frac{f(5) - f(3)}{5-3}$$

Since $-1 \le f'(c) < 2$ this means

$$-1 \le \frac{f(5) - f(3)}{5-3} \le 2$$

$$-1 \le \frac{f(5) - 7}{2} \le 2 \quad | \cdot 2$$

$$-2 \le f(5) - 7 \le 4 \quad | + 7$$

$$5 \le f(5) \le 11$$

5. (6pts) Find the absolute minimum and maximum values for the function $f(x) = x^3e^x$ on the interval $[-4, 1]$.

$f'(x) = 3x^2e^x + x^3e^x$

$= e^x(x^3+3x^2)$

$e^x(x^3+3x^2)=0 \quad | \div e^x$, never zero

$x^2(x+3)=0$

$x=0$ or $x=-3$

x	$f(x)$	
0	0	
-3	$(-3)^3e^{-3} = -1.34$	$\leftarrow$ abs. min.
-4	$(-4)^3e^{-4} = -1.17$	
1	$1e^1 = 2.71$	$\leftarrow$ abs max

6. (6pts) Sketch the graph of a function defined on all reals satisfying:

$f'(x) > 0$ if $x < 0$

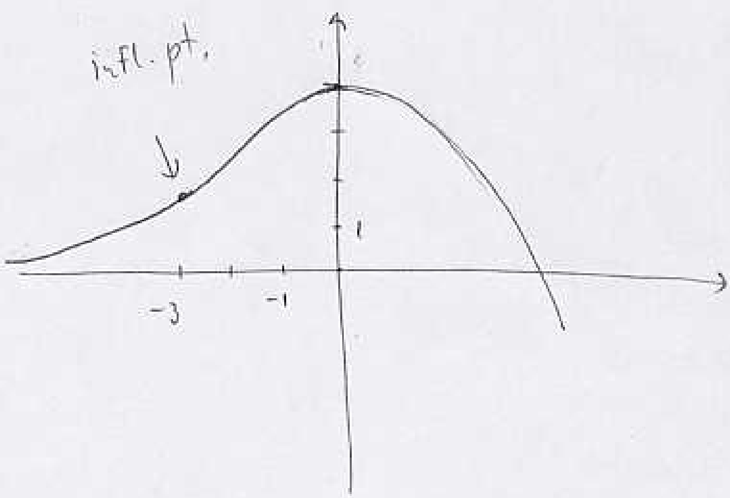
$f'(x) < 0$ if $x > 0$

$f'(0) = 0$, $f(0) = 4$

$f''(x) > 0$ if $x < -3$

$f''(x) < 0$ if $x > -3$

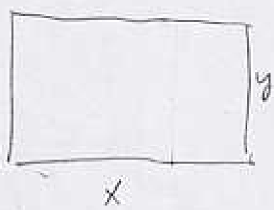
$\lim_{x\to-\infty} f(x) = 0$.



f incr. on $(-\infty, 0)$
decr on $(0, \infty)$

f conc up on $(-\infty, 3)$
conc dn on $(-3, \infty)$

7. (8pts) Farmer Tom wants to fence in a rectangular area of 5km^2. What dimensions of the rectangle minimize the cost of the fence? Verify that your dimensions indeed give you a minimal cost.



$xy = 5$ so $y = \frac{5}{x}$

$L = 2(x+y) = 2(x + \frac{5}{x})$

Job: minimize L on $(0, \infty)$

$L'(x) = 2(1 - \frac{5}{x^2})$

$2(1 - \frac{5}{x^2}) = 0$

$1 = \frac{5}{x^2}$

$x^2 = 5$

$x = \pm\sqrt{5}$

$-\sqrt{5}$ not in $(0, \infty)$

$L''(x) = 2 \cdot (-1) 5 \cdot (-2) x^{-3}$

$= \frac{20}{x^3}$

$L''(\sqrt{5}) > 0$

so L has a local maximum at $\sqrt{5}$. It is an absolute max, since the interval is open and there is only one local max.

Bonus. (5pts) Suppose f is a function that is positive ($f(x) > 0$) and concave up on $(-1, 1)$.
a) Show that $g(x) = [f(x)]^2$ is concave up on $(-1, 1)$.
b) Find a an example of a function f that is concave up, yet negative on $(-1, 1)$ for which $g = f^2$ is concave down. (Hint: think of a simple function!)

a) $f(x) > 0$

$f''(x) > 0$

Find g'':

$g'(x) = 2f(x)f'(x)$

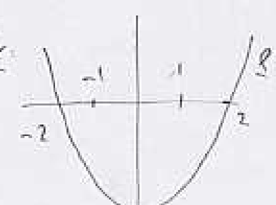
$g''(x) = 2(f'(x)f'(x) + f(x)f''(x))$

$= 2(\underbrace{[f'(x)]^2}_{\text{always } \geq 0} + \underbrace{f(x)}_{>0}\underbrace{f''(x)}_{>0})$

by assumption on $(-1,1)$

Since $g''(x) > 0$, g is concave up.

b) Ex



$f(x) = x^2 - 4$

$g(x) = (x^2 - 4)^2$