Fall '04/MAT 250/Exam 2 **Name:** Solution *Show all your work.*

Differentiate and simplify where appropriate:

1. (4pts) $\frac{d}{dx}(3x^4 - 2^x + 5\sqrt[3]{x} + e^2) = 12x^3 - \ln 2 \cdot 2^x + \frac{5}{3}x^{-\frac{2}{3}}$

$x^{\frac{1}{3}}$ constant

2. (5pts) $\frac{d}{dx}\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$= \frac{e^{2x} - 2 + e^{-2x} - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$

3. (5pts) $\frac{d}{d\theta}\sin^4\theta\cos^4\theta = 4\sin^3\theta\cos\theta\cos^4\theta + \sin^4\theta \cdot 4\cos^3\theta(-\sin\theta)$

$= 4\sin^3\theta\cos^5\theta - 4\sin^5\theta\cos^3\theta$

$= 4\sin^3\theta\cos^3\theta(\cos^2\theta - \sin^2\theta)$

4. (5pts) $\frac{d}{dx}\left(2x\arctan x - \ln(1+x^2)\right) = 2\arctan x + 2x\frac{1}{1+x^2} - \frac{1}{1+x^2}\cdot 2x$

$= 2\arctan x$

5. (5pts) Use logarithmic differentiation to find $\dfrac{d}{dx}x^{\frac{1}{x}}$.

$y = x^{\frac{1}{x}} \quad | \ln$

$\ln y = \ln x^{\frac{1}{x}} = \frac{1}{x}\ln x \quad \Big| \frac{d}{dx}$

$\dfrac{y'}{y} = \dfrac{\frac{1}{x}\cdot x - \ln x}{x^2}$

$y' = x^{\frac{1}{x}} \cdot \dfrac{1 - \ln x}{x^2}$

6. (3pts) Find $D^{127}\cos 4x$.

$D^{127}\cos 4x = 4^{127}\sin 4x$

$y = \cos 4x$

$y' = 4\sin 4x$

$y'' = -4^2\cos 4x$

$y''' = -4^3\sin 4x$

$y^{(4)} = -4^4\cos 4x$

after 124 derivatives have $\cos 4x$

Taking 3 more derivatives gets you $\sin 4x$

7. (6pts) Consider the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$.

a) Use implicit differentiation to find the equation of the tangent line to this curve at the point $(-1, 2\sqrt{3})$.

b) Draw a picture of the ellipse and the tangent line and make sure your answer in a) agrees with what you see in the picture.

a) $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1 \quad \Big| \frac{d}{dx}$

$\dfrac{2x}{4} + \dfrac{2yy'}{16} = 0$

$\dfrac{yy'}{8} = -\dfrac{x}{2}$

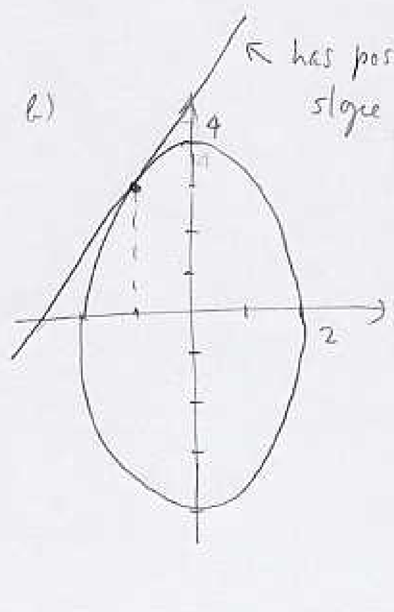
$y' = -\dfrac{4x}{y}$

At $(-1, 2\sqrt{3})$: $y' = -\dfrac{4\cdot(-1)}{2\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

$y - 2\sqrt{3} = \dfrac{2}{\sqrt{3}}(x+1)$

$y = \dfrac{2}{\sqrt{3}}x + \dfrac{2}{\sqrt{3}} + 2\sqrt{3}$

b)



has positive slope

8. (4pts) The radius r of a disk is increasing.
a) At what rate with respect to the radius is the area of the disk increasing when $r = 5$cm?
b) Approximate by how much the area changes if r changes from 5 to 5.1 centimeters.

a) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dr}(5) = 10\pi \ cm^2/cm$

b) $\Delta A \approx A'(r)\Delta r$ $\qquad \Delta r = 0.1$

$\Delta A \approx 10\pi \cdot 0.1$

$\approx \pi \ cm^2$

9. (6pts) A jet-propelled head of cabbage is moving so that its position function is given by $s(t) = t^3 - 9t^2 + 30t + 113$, where s is in feet, t in seconds. Find the acceleration at the moment(s) when velocity equals 6ft/s.

$s(t) = t^3 - 9t^2 + 30t + 113$

$v(t) = 3t^2 - 18t + 30$

$a(t) = 6t - 18$

$3t^2 - 18t + 30 = 6$

$3t^2 - 18t + 24 = 0 \quad | \div 3$

$t^2 - 6t + 8 = 0$

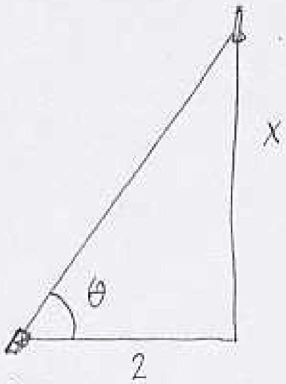
$(t-4)(t-2) = 0$

$t = 2, 4$

$a(2) = 6 \cdot 2 - 18 = -6 \ ft/s^2$

$a(4) = 24 - 18 = 6 \ ft/s^2$

10. (7pts) A camera is positioned 2km away from the launch pad of a rocket. After launch, the rocket rises along a straight vertical line while the camera tracks it. Suppose an instrument on the camera tells you that when the angle of elevation of the camera is $\pi/4$, the angle of elevation is increasing at rate 0.01 radians/second. How fast is the rocket rising at that moment? (The angle of elevation is the angle between the line of sight and the horizontal. You may assume the camera is lying on the ground.)



camera

$$\tan\theta = \frac{X}{2} \quad \Big| \frac{d}{dt}$$

$$\sec^2\theta \cdot \theta' = \frac{1}{2}x'$$

$$x' = 2\sec^2\theta \cdot \theta'$$

$$x' = \frac{2}{\cos^2\frac{\pi}{4}} \cdot 0.01 = \frac{2}{\left(\frac{1}{\sqrt{2}}\right)^2} \cdot 0.01 = 0.04 \text{ km/s}$$

know $\frac{d\theta}{dt} = 0.01$ when $\theta = \frac{\pi}{4}$

Need $\frac{dx}{dt}$ when $\theta = \frac{\pi}{4}$

Bonus. (5pts) Find the first four derivatives of $y = \frac{1}{x}$. Use the derivatives to find a pattern and say what the n-th derivative is.

$$y = \frac{1}{x} = x^{-1}$$

$$y' = (-1)x^{-2}$$

$$y'' = (-1)(-2)x^{-3}$$

$$y''' = (-1)(-2)(-3)x^{-4}$$

$$y^{(4)} = (-1)(-2)(-3)(-4)x^{-5}$$

$$y^{(n)} = \overbrace{(-1)(-2)\ldots(-n)}^{n \text{ minuses}}x^{-n-1}$$

$$= (-1)^n 1 \cdot 2 \cdot \ldots \cdot n\, x^{-n-1}$$

$$= \frac{(-1)^n n!}{x^{n+1}}$$